72. Cech Cohomology of Foliations and Transverse Measures

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(Communicated by Heisuke HIRONAKA, M. J. A., June 15, 1982)

§1. **Introduction.** For the holonomy groupoid Γ of a foliation (M, F), the transverse measure with modulus in the non-commutative integration theory [1] of A. Connes is an extension of the ordinary concept of transverse measures for the foliation. We will find in Theorem 1 a necessary and sufficient condition for the modulus δ in order that a faithful transverse measure in the Lebesgue measure class with this modulus δ exists. The condition is that δ belongs to the canonical Cech cohomology class of the foliation.

We shall define the associated foliation $(\tilde{M}, \tilde{F})$ for a given foliation (M, F) and show in Theorem 2 that the canonical Cech cohomology class of $(\tilde{M}, \tilde{F})$ vanishes and the von Neumann algebra associated with $(\tilde{M}, \tilde{F})$ is the crossed product of the same for (M, F) by its modular action.

§2. **Definitions.** We mainly follow the notations and the terminology in [1]. We assume that the holonomy groupoid Γ is a Hausdorff space.

Let us define a sheaf C_F^∞ on M, whose sections are real-valued C^∞-functions constant along leaves. More precisely, for each open subset U of M, the sections of C_F^∞ over U is given by

(1) $$C_F^\infty(U)=\{f \in C^\infty(U)\,;\, Xf=0 \text{ for } X \in TF\}.$$

Analogously we define a sheaf L_F on M, whose sections are Lebesgue measurable functions constant along leaves. Its precise definition is as follows: Set

(2) $\tilde{L}_F(U)=\{f\,;\, f$ is a real-valued Lebesgue measurable function, constant along plates in $U\}$.

Here we don't distinguish two almost equal functions. $\tilde{L}_F(U)$ forms a presheaf on M and L_F is defined to be the sheaf generated by $\tilde{L}_F$.

Now we associate a cohomology class $c(F)$ in $H^1(M, C_F^\infty)$ with the foliation (M, F). Take an open covering $\mathcal{U}=\{U_\alpha\}$ of M by foliated charts. We denote the transversal coordinates in U_α by q_α^i. Let $c_{\alpha\beta}$ be an element of $C_F^\infty(U_\alpha \cap U_\beta)$ defined by

(3) $$c_{\alpha\beta}=\log\left|\det\left(\frac{\partial q_\beta^i}{\partial q_\alpha^j}\right)\right|.$$

Then $c_{\alpha\beta}$ satisfies the cocycle condition

$$(4) \qquad c_{\alpha\beta}+c_{\beta\gamma}+c_{\gamma\alpha}=0 \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$

and represents a cohomology class in $H^1(\mathcal{U}, C_F^\infty)$. Taking the inductive limit with respect to the covering, we obtain a Cech cohomology class $c(F)$ in $H^1(M, C_F^\infty)$.

Since C_F^∞ can be regarded as a subsheaf of L_F, there is a canonical homomorphism of $H^1(M, C_F^\infty)$ into $H^1(M, L_F)$. The image of $c(F)$ under this homomorphism is denoted by $\tau(F)$.

Finally we define a cohomology group associated with the holonomy groupoid Γ. Let $Z(\Gamma)$ be the set of Lebesgue measurable homomorphisms of Γ into $\boldsymbol{R}_+$ (the set of positive real numbers). As before, two almost equal homomorphisms are regarded as the same. Let $B(\Gamma)$ be a subset of $Z(\Gamma)$ whose element h is of the form

$$(5) \qquad h(\gamma)=f(r(\gamma))f(s(\gamma))^{-1} \qquad \text{for } \gamma \in \Gamma$$

where f is a R_+-valued Lebesgue measurable function on the unit space $\Gamma^{(0)}$. We set $H(\Gamma)=Z(\Gamma)/B(\Gamma)$.

§ 3. Obstruction to the existence of transverse measures.

Proposition. *We have a canonical isomorphism:*

$$H(\Gamma) \cong H^1(M, L_F).$$

Sketch of the proof. Let $[h] \in H(\Gamma)$ with $h \in Z(\Gamma)$. We want to construct the corresponding element in $H^1(M, L_F)$. Take an open covering $\mathcal{U}=\{U_\alpha\}$ of M and $\boldsymbol{R}_+$-valued measurable function h_α on U_α such that

$$h=(h_\alpha \circ r)(h_\alpha \circ s)^{-1} \qquad \text{on } \Gamma_\alpha$$

where Γ_α is the holonomy groupoid associated with the foliation $(U_\alpha, F|_{U_\alpha})$. Set $c_{\alpha\beta}=\log h_\alpha|_{U_\alpha \cap U_\beta}-\log h_\beta|_{U_\alpha \cap U_\beta}$. Then $c_{\alpha\beta} \in L_F(U_\alpha \cap U_\beta)$ and $\{c_{\alpha\beta}\}$ satisfies the cocycle condition and defines a cohomology class in $H^1(\mathcal{U}, L_F)$. Letting the covering $\mathcal{U}$ finer and finer, we obtain a cohomology class in $H^1(M, L_F)$. Now one can prove that this correspondence is well-defined and gives rise to an isomorphism between $H(\Gamma)$ and $H^1(M, L_F)$.

Remark. In the following we identify $H(\Gamma)$ with $H^1(M, L_F)$ by the above isomorphism.

Let δ be an element of $Z(\Gamma)$. For a nowhere vanishing C^∞-density D of TF, $\nu \equiv s^*D$ defines a faithful transverse function on Γ ([1] p. 119) and we have a 1–1 correspondence between transverse measure Λ on Γ with modulus δ and δ-symmetric measure Λ_ν on $\Gamma^{(0)}$ ([1] Theorem II. 3, [2] Theorem 10). A transverse measure Λ is called to be in Lebesgue class if the measure Λ_ν is equivalent to a measure defined by a nowhere vanishing C^∞-density on $\Gamma^{(0)}$. Now the following theorem is an easy consequence of the definitions.

Theorem 1. *Γ has a Lebesgue class transverse measure with modulus δ if and only if the cohomology class $[\delta]$ of δ in $H(\Gamma)$ is equal to $\tau(F)$.*

§4. **Crossed products for foliations.** Given a foliation (M, F), we denote by $W^*(M, F)$ the von Neumann algebra of random operators on the holonomy groupoid (with Lebesgue measure class) ([1] p. 84). We shall construct a new foliation $(\tilde{M}, \tilde{F})$ for which the cohomology class $c(\tilde{F})$ vanishes and the associated von Neumann algebra $W^*(\tilde{M}, \tilde{F})$ is isomorphic to the crossed product of $W^*(M, F)$ by its modular action.

First note that we have the de Rham type isomorphism (see [5] for example):

(6) $$H^1(M, C_F^\infty) \cong \{\theta ; \theta \text{ is a } C^\infty\text{-section of } T^*F \text{ such that } d_F\theta = 0\} / \{d_F f ; f \text{ is a } C^\infty\text{-function on } M\},$$

where d_F denotes the exterior derivative along leaves. Let θ be a d_F-closed 1-form which represents the cohomology class $c(F)$ by the above isomorphism. Set $\tilde{M} = M \times \boldsymbol{R}$ and $E = \{(X, v) \in T(M \times \boldsymbol{R}) ; v = \theta(X), X \in TF\}$. Then E is an involutive subbundle of $T\tilde{M}$ and defines a foliation $\tilde{F}$ in $\tilde{M}$. Let π be the canonical projection of $\tilde{M}$ onto M. The foliation $(\tilde{M}, \tilde{F})$ has the following properties:

(i) $\pi : \tilde{M} \to M$ has a structure of principal $\boldsymbol{R}$-bundle.

(ii) $\pi_*(T\tilde{F}) = TF$.

(iii) $\tilde{F}$ is $\boldsymbol{R}$-invariant.

(iv) $b(\tilde{M}, \tilde{F}) = c(F)$,

where $b(\tilde{M}, \tilde{F})$ is a cohomology class in $H^1(M, C_F^\infty)$ associated with the foliated $\boldsymbol{R}$-bundle $(\tilde{M}, \tilde{F})$ defined as follows: Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M and $\rho_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \boldsymbol{R}$ be a local trivialization of $\boldsymbol{R}$-bundle $\tilde{M}$. Let $\{g_{\alpha\beta}\}$ be the family of transition functions ($g_{\alpha\beta}$ is a C^∞-function on $U_\alpha \cap U_\beta$). By (ii) and (iii), there is a 1-form θ_α on U_α such that

(7) $$\rho_{\alpha *}(TF) = \{(X, \theta_\alpha(X)) \in T(U_\alpha \times \boldsymbol{R}) ; X \in TF\}$$

and we have $\theta_\alpha - \theta_\beta = d_F g_{\alpha\beta}$. Since θ_α is d_F-closed (by the involutivity of $T\tilde{F}$), we can find $f_\alpha \in C^\infty(U_\alpha)$ such that $\theta_\alpha = d_F f_\alpha$. The relation $\theta_\alpha - \theta_\beta = d_F g_{\alpha\beta}$ implies that $f_\alpha - f_\beta - g_{\alpha\beta} \in C_F^\infty(U_\alpha \cap U_\beta)$, and $\{f_\alpha - f_\beta - g_{\alpha\beta}\}$ defines a cohomology class in $H^1(\mathcal{U}, C_F^\infty)$. Taking the inductive limit for the covering $\mathcal{U}$, we get the cohomology class $b(\tilde{M}, \tilde{F})$ in $H^1(M, C_F^\infty)$.

Conversely one can show that a foliation $(\tilde{M}, \tilde{F})$ satisfying above conditions (i)–(iv) is isomorphic to the one constructed from (M, F) at the beginning of this section.

Summarizing this section, we have the following:

Theorem 2.

(i) $c(\tilde{F}) = 0$.

(ii) $W^*(\tilde{M}, \tilde{F}) \cong W^*(M, F) \times_\sigma \boldsymbol{R}$,

where σ is a modular action for $W^*(M, F)$.

Corollary. *$(\tilde{M}, \tilde{F})$ has a Lebesgue class transverse measure with trivial modulus.*

Remark. The $\boldsymbol{R}$-action (as principal $\boldsymbol{R}$-bundle) on $\tilde{M}$ leaves $\tilde{F}$ invariant and it induces an action ϕ of $\boldsymbol{R}$ on $W^*(\tilde{M}, \tilde{F})$ which coincides with the dual action of the modular action. The crossed product of $W^*(\tilde{M}, \tilde{F})$ by ϕ is also expressed by foliation. In fact, let $\tilde{\tilde{M}} = \tilde{M}$ $(= M \times \boldsymbol{R})$ and define a foliation $\tilde{\tilde{F}}$ in $\tilde{\tilde{M}}$ so that leaves in $\tilde{\tilde{F}}$ are of the form $\mathcal{L} \times \boldsymbol{R}$ with $\mathcal{L}$ in F. Then one can show that the holonomy groupoid of $(\tilde{\tilde{M}}, \tilde{\tilde{F}})$ is isomorphic to the semidirect product of $\tilde{\Gamma}$ (the holonomy groupoid of $(\tilde{M}, \tilde{F})$) by ϕ (see [1] p. 65, [3] for the definition of semidirect product) and hence $W^*(\tilde{\tilde{M}}, \tilde{\tilde{F}})$ is isomorphic to the crossed product of $W^*(\tilde{M}, \tilde{F})$ by ϕ. This provides the geometrical version of Takesaki's duality ([4]) $W^*(\tilde{M}, \tilde{F}) \times_\phi \boldsymbol{R} \cong W^*(M, F) \otimes \mathcal{B}(L^2(\boldsymbol{R}))$.

References

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