68. *Nonlinear Perron-Frobenius Problem*

An Extension of Morishima's Theorem

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1. Introduction. In connection with the discrete version of Boltzmann equation and Volterra's ecology equation, many researches have been done on the homogeneous quadratic differential equation of the form:

$$\frac{du_j}{dt}=\mathrm{A}_j(u_1,\cdots,u_n),\qquad j=1,\cdots,n$$

where each $A_j(u)$ is a quadratic form (cf. Jenks [5], Carleman [6]).

The important first approach to this type of equations is to find their positive equilibrium points, i.e., the equilibrium points whose coordinates are all positive.

The author studied the special case of this equation:

$$\frac{du_j}{dt}=A_j(u_1,\cdots,u_n)-\mu u_j^2,\qquad j=1,\cdots,n$$

where each $A_j(u)$ is a quadratic form with non-negative coefficients. In this case the search of positive equilibrium points is converted to the eigen-value problem of the form:

$$H(u)=\lambda u$$

where $H_j(u)=[A_j(u)]^{1/2}$ and $\lambda=\mu^{1/2}$.

The author was later informed that the theoretical economists had done a great contribution to this type of problems in connection with the balanced growth problem (cf. Morishima [1], Nikaido [2]), and that in the case of Banach spaces, there is a book of Krasnoselskii [4].

But the newly introduced notion of non-sectionality is wider than the indecomposability defined by the economists, and not yet known to the researchers of the Boltzmann equation. So the author hopes that it is useful for both mathematicians and economists.

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2. Definitions and Notations. We use vector inequalities. $x\leqq y$ means $x_j\leqq y_j$ (for all $j=1,\cdots,n$), and $x<y$ implies $x_j<y_j$ (for all $j=1,\cdots,n$). And $x\leq y$ implies $x\leqq y$ and $x\neq y$.

$H(x)$ is called a non-negative non-decreasing homogeneous transformation of degree one if it satisfies:

1) $H(x)$ is a mapping from $[0, \infty)\times\cdots\times[0, \infty)$ into itself.
2) $H(x)$ is continuous in this closed domain.
3) $H(x)\leqq H(y)$ for all $x\leqq y$.
4) For all $\mu\geqq 0$, $H(\mu x)=\mu H(x)$. (Therefore $H(0)=0$.)

For the simplicity, we simply call $H(x)$ a homogeneous transformation. And our problem is to find non-negative λ and x satisfying

$$H(x)=\lambda x.$$

We define the principal notion of the non-sectionality and refer to another notion of the indecomposability for the sake of comparison. (For the indecomposability, cf. Morishima [1].) As easily seen, the non-sectionality is wider than the indecomposability.

Definition 1. A homogeneous transformation $H(x)$ is called non-sectional if it satisfies the following: For any given proper partition $\Theta\cup\Omega=\{1, \cdots, n\}$ (i.e. $\Theta\cap\Omega=\phi$, $\Theta\neq\phi$, $\Omega\neq\phi$), there exists always $\bar{\omega}\in\Omega$ satisfying the next two conditions at the same time.

1) $H_{\bar{\omega}}(x)<H_{\bar{\omega}}(\bar{x})$ for all x, $\bar{x}$ such that $x_\theta<\bar{x}_\theta$ for all $\theta\in\Theta$, $0<x_\omega=\bar{x}_\omega$ for all $\omega\in\Omega$.

2) $\lim_{x_\theta\to\infty} H_{\bar{\omega}}(x)=\infty$ (for all $\theta\in\Theta$), where $x_\omega>0$ (for all $\omega\in\Omega$) are fixed.

Definition. 2. A homogeneous transformation is called indecomposable if it satisfies the following: For any given proper partition $\Theta\cup\Omega=\{1, \cdots, n\}$, there exists always $\omega\in\Omega$ satisfying the next condition.

1) $H_{\bar{\omega}}(x)<H_{\bar{\omega}}(\bar{x})$ for all x, $\bar{x}$ such that $x_\theta<\bar{x}_\theta$ for all $\theta\in\Theta$, $0\leqq x_\omega=\bar{x}_\omega$ for all $\omega\in\Omega$.

3. Theorems (we shall omit all the proofs to save the space). The following fact is known (cf. Morishima [1]).

Theorem 1. *Any homogeneous transformation $H(x)$ has a finite number* (at least one) *of non-negative eigen-values which correspond to non-negative eigen-vectors.*

A sufficient condition to have a strictly positive eigenvector is given as follows (cf. Morishima [1]).

Theorem 2. *If $H(x)$ is indecomposable, then it has only one non-negative eigen-vector, moreover it is positive.*

But this condition is too stringent. Let us consider a specific example of non-sectional transformation:

$$H_1(x, y, z)=x^{1/2}(x+y+z),$$
$$H_2(x, y, z)=y^{1/2}(x+y+z),$$
$$H_3(x, y, z)=z^{1/2}(x+y+z).$$

This $H(x)$ is not indecomposable because, for instance,

$$0=H_3(0, 0, 0)=H_3(1, 1, 0)$$

while $\Theta=\{1,2\}$, $\Omega=\{3\}$. Yes it has a unique positive eigen-vector $(1,1,1)^T$, i.e.,

$$H(1,1,1)=3^{1/2}(1,1,1)^T.$$

As easily shown, this $H(x)$ has $(1,0,0)^T$, $(1,1,0)^T$ also as eigen-vectors whose eigen-values are smaller than $3^{1/2}$, which is different from the case of indecomposable transformations. We can explain this fact by the following

Theorem 3. *If $H(x)$ is non-sectional, then the eigen-vector corresponding to the maximal eigen-value is positive and unique.*

We shall denote the maximal eigen-value of $H(x)$ by $\lambda_0(H)$ or λ_0 from now on. We can prove another

Theorem 4. *If $H(x)\le \bar{H}(x)$ (for all $x>0$) and $\bar{H}(x)$ is non-sectional, then $\lambda_0(H)<\lambda_0(\bar{H})$.*

Now we look into resolvent problem. We consider the non-negative solutions of the following resolvent equation.

$$(1) \qquad \lambda x-H(x)=c \qquad (c\geqq 0).$$

We can prove the following

Theorem 5. *Suppose $H(x)$ is a non-sectional homogeneous transformation. If $\lambda>\lambda_0(H)$, then there exists a unique solution for (1) with $c>0$. We denote these solutions by $R_\lambda(c)$. This function of c is continuous in $(0,\infty)\times\cdots\times(0,\infty)$ and has a continuous extension in $[0,\infty)\times\cdots\times[0,\infty)$ which is still a solution of (1) in this closed domain. Moreover,*

1) *$R_\lambda(c)>0$ for all $c\ge 0$ and $R_\lambda(0)=0$,*

2) *for $c>0$ and $c=0$, $R_\lambda(c)$ is the unique solution of (1). For $c\ge 0$, there exists no other positive solution than $R_\lambda(c)$,*

3) *for $0\leqq c\leqq \bar{c}$, $R_\lambda(c)\leqq R_\lambda(\bar{c})$. For $0\leqq c\le \bar{c}$, $R_\lambda(c)<R_\lambda(\bar{c})$.*

Conversely, if there exists a positive solution for the resolvent equation (1) with $c\ge 0$, then $\lambda>\lambda_0(H)$.

Let us investigate general transformations, i.e., sectional transformations, from now on.

We put some assumptions on the transformations.

α) *$H(x)$ is real-analytic in $(0,\infty)\times\cdots\times(0,\infty)$, and continuous in $[0,\infty)\times\cdots\times[0,\infty)$.*

β) *For any given proper partition $\Theta\cup\Omega=\{1,\cdots,n\}$, if there should exist ϖ which satisfies the first condition of the non-sectionality, then ϖ must also satisfy the second condition.*

Theorem 6. *There exists a cannonical form of $H(x)$ with the partition $J_1\cup\cdots\cup J_\mu\cup J_{\mu+1}\cup\cdots\cup J_\nu=\{1,\cdots,n\}$ such that*

$$H_{J_k}(x)\equiv H_{J_k}(x_{J_k},0) \qquad \text{for all } k=1,\cdots,\mu$$

$$H_{J_k}(x)\equiv F_{J_k}(x)+H_{J_k}(x_{J_k},0) \qquad \text{for all } k=\mu+1,\cdots,\nu$$

where each $H_{J_k}(x_{J_k},0)$ is non-sectional as transformation of x_{J_k}, $F_{J_k}(0,$

$0, \cdots, 0, x_{J_k}, x_{J_{k+1}}, \cdots, x_{J_\nu}) \equiv 0$ *but* $F_{J_k}(x) \not\equiv 0$. *Moreover the partition* $J_1, \cdots, J_\nu$ *is unique.*

Remark. x_{J_k} is a shorter notation of x_j (for all $j \in J_k$), and $(x_{J_k}, 0)$ is a vector where $x_j = 0$ (for all $j \notin J_k$), $H_{J_k}(x)$ is a shorter notation of $H_j(x)$ (for all $j \in J_k$).

By using this cannonical form, we can formulate the following theorem which characterizes the maximal eigen-value of sectional homogeneous transformation.

Theorem 7. *Let* $H(x)$ *be written in a cannonical form with* $J_1 \cup \cdots \cup J_\mu \cup J_{\mu+1} \cup \cdots \cup J_\nu = \{1, \cdots, n\}$. *Then we can express its maximal eigen-value by*

$$\lambda_0(H) = \max_{1 \leqq k \leqq \nu} \{\lambda_0(H_{J_k}(x_{J_k}, 0))\}.$$

Finally we give a necessary and sufficient condition to have a positive eigen-vector under the assumptions α) and β).

Theorem 8. *Let* $H(x)$ *be written in a cannonical form with*

$$J_1 \cup \cdots \cup J_\mu \cup J_{\mu+1} \cup \cdots \cup J_\nu = \{1, \cdots, n\}.$$

Then $H(x)$ *has a positive eigen-vector if and only if it satisfies the following two conditions at the same time*:

1) $\lambda_0 H_{J_k}(x_{J_k}, 0)) = \lambda_0(H)$ *for all* $k = 1, \cdots, \mu$

2) $\lambda_0 H_{J_k}(x_{J_k}, 0)) < \lambda_0(H)$ *for all* $k = \mu + 1, \cdots, \nu$.

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