66. *On the Microlocal Structure of a Regular Prehomogeneous Vector Space Associated with Spin(10)×GL(3)*

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Let ρ_1 be the even half-spin representation of the spin group *Spin* (10). Its representation space $V(16)$ is spanned by 1, e_ie_j, $e_ke_le_se_t$ ($1\leqq i<j\leqq 5$, $1\leqq k<l<s<t\leqq 5$) over C. Define e_i^* by $e_ie_i^*=e_1e_2e_3e_4e_5$, i.e., $e_1^*=e_2e_3e_4e_5$, $e_2^*=-e_1e_3e_4e_5$, etc. Let $\rho=\rho_1\otimes\Lambda_1$ be the representation of the group $G=Spin(10)\times GL(3)$ on $V=V(16)\otimes V(3)$ where Λ_1 denotes the standard representation of $GL(3)$ on $V(3)$. Then the triplet (G,ρ,V) is an irreducible regular prehomogeneous vector space ([1]). There exists a unique relatively invariant irreducible polynomial $f(x)$ of (G,ρ,V) with $\deg f(x)=12$. In this article, we give the orbital decomposition of (G,ρ,V) and the b-function $b(s)$ of the relative invariant $f(x)$ by constructing the holonomy diagram (see [2], [3]). All other irreducible regular P.V.'s have been already treated in [2]–[6].

§1. **The orbits.** Let ρ^* be the contragredient representation of ρ on the dual space V^* of V. We identify the cotangent bundle T^*V with $V\times V^*$. Let S (resp. S^*) be a G-orbit in V (resp. V^*), Λ (resp. Λ^*) the Zariski-closure of the conormal bundle of S (resp. S^*). Then Λ and Λ^* are subsets of $V\times V^*$. If $\Lambda=\Lambda^*$, we say that S and S^* are dual orbits of each other. Let W be the Zariski-closure of $\{(x, s\,\mathrm{grad}\log f(x))\in V\times V^*;\ f(x)\neq 0,\ s\in C\}$ in $V\times V^*$. It is known that if Λ has a Zariski-dense G-orbit, i.e., G-prehomogeneous, and $\Lambda\subset W$, then the micro-differential equations $\mathfrak{M}=\mathcal{E}f^s$ is a simple holonomic system near a generic point of Λ, and its order $\mathrm{ord}_\Lambda f^s$ is uniquely determined (see [2]). Since G is reductive, we have $(G,\rho,V)\cong(G,\rho^*,V^*)$ and we identify V^* with V.

Let S_{ij}^k be the i-codimensional G-orbit in V with the j-codimensional dual orbit such that its isotropy subgroup has a k-dimensional unipotent part. We denote by Λ_{ij}^k the Zariski-closure of the conormal bundle of S_{ij}^k. In Table I, N.P. (resp. $\not\subset W$) implies that Λ_{ij}^k is not G-prehomogeneous (resp. $\Lambda_{ij}^k\not\subset W$). In the case that Λ_{ij}^k is G-prehomogeneous and $\Lambda_{ij}^k\subset W$, the order $\mathrm{ord}_\Lambda f(x)^s$ of the simple holonomic system $\mathfrak{M}=\mathcal{E}f^s$ on $\Lambda=\Lambda_{ij}^k$ is given in Table I.

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Theorem 1. *The triplet* $(Spin(10)\times GL(3)$, *half-spin rep.* $\otimes \Lambda_1$, $V(16)\otimes V(3))$ *has thirty-two orbits given in Table I.*

Remark 1. We identify $V=V(16)\otimes V(3)$ with $V(16)\oplus V(16)\oplus V(16)$. The isotropy subgroups are given up to local isomorphism. In general, $U(n)$ (resp. G_a) denotes an n-dimensional unipotent group (resp. the one-dimensional additive group). In Table I, $\times$ (resp. $\cdot$) means the direct product (resp. semi-direct product).

Remark 2. The orbital decomposition was first tried by H. Kawahara (see [7]). Although he missed the orbit $S^9_{11,15}$, his method is effective for the complete orbital decomposition.

Remark 3. The prehomogeneity of the triplet (G, ρ, V) is also obtained from that of other triplets as follows. Since $(Spin(10)\times GL(2), \rho_1\otimes\Lambda_1, V(16)\otimes V(2))$ and $(G_2\times GL(2), \Lambda_2\otimes\Lambda_1, V(7)\otimes V(2))$ are P.V.'s (see [1]), one can see easily that the triplet $((GL(1)\times Spin(10))\times GL(14), (\Lambda_1\otimes 1+1\otimes\rho_1)\otimes\Lambda_1, (V(1)\oplus V(16))\otimes V(14))$ is a P.V., and so is its castling transform $((GL(1)\times Spin(10))\times GL(3), (\Lambda_1\otimes 1+1\otimes\rho_1)\otimes\Lambda_1, (V(1)\oplus V(16))\otimes V(3))$. In particular, $(Spin(10)\times GL(3), \rho_1\otimes\Lambda_1, V(16)\otimes V(3))$ is a P.V.

Table I

The orbits		Representative points	Isotropy subgroups	Order	The dual orbits
(1)	$S^0_{0,48}$	$(1+e_1^*, e_1e_2+e_2^*, e_2e_3+e_3^*)$	$SL(2)\times SL(2)$	0	$S^0_{48,0}$
(2)	$S^7_{1,27}$	$(1+e_1^*, e_1e_2+e_3e_4, e_1e_3+e_4e_5)$	$GL(1)^2\cdot U(5)$	$-s-1/2$	$S^{17}_{27,1}$
(3)	$S^5_{3,19}$	$(1+e_1^*, e_1e_2+e_3e_4, e_1e_3+e_3^*)$	$(GL(1)\times SL(2))\cdot U(5)$	$-2s-3/2$	$S^{17}_{19,3}$
(4)	$S^4_{3,35}$	$(1+e_1^*, e_2e_3+e_2^*, e_1e_2)$	$(GL(1)^2\times SL(2))\cdot U(4)$	$\not\subset W$	$S^{12}_{35,3}$
(5)	$S^7_{5,15}$	$(1+e_1^*, e_1e_2+e_2^*, e_2e_3+e_4^*)$	$(GL(1)\times SL(2))\cdot U(7)$	$\not\subset W$	$S^{15}_{15,5}$
(6)	$S^7_{5,23}$	$(1+e_1^*, e_1e_2+e_3e_4, e_4e_5+e_5^*)$	$(GL(1)\times SL(2))\cdot G_a^7$	N.P.	$S^{14}_{23,5}$
(7)	$S^9_{6,14}$	$(1+e_1^*, e_2e_3+e_2^*, e_1e_4)$	$GL(1)^3\cdot U(9)$	$-5s-12/2$	$S^7_{14,6}$
(8)	$S^2_{6,22}$	$(1, e_1^*, e_1e_2+e_2^*)$	$(GL(1)^2\times SL(3))\cdot U(2)$	$\not\subset W$	$S^7_{22,6}$
(9)	$S^{10}_{7,17}$	$(1, e_1e_2+e_1^*, e_1e_3+e_4^*)$	$GL(1)^3\cdot U(10)$	N.P.	$S^{14}_{17,7}$
(10)	$S^8_{7,23}$	$(1+e_1^*, e_1e_2+e_2^*, e_2e_3)$	$(GL(1)^2\times SL(2))\cdot U(8)$	$-3s-8/2$	$S^{16}_{23,7}$
(11)	$S^{11}_{8,8}$	$(1, e_1e_2+e_3e_4, e_1e_3+e_1^*)$	$GL(1)^3\cdot U(11)$	$-6s-17/2$	self-dual
(12)	$S^9_{9,9}$	$(1, e_1^*, e_1e_2+e_3^*)$	$(GL(1)^3\times SL(2))\cdot U(9)$	$-6s-15/2$	self-dual
(13)	$S^{10}_{10,10}$	$(1, e_1^*, e_1e_2+e_3e_4)$	$(GL(1)^3\times SL(2))\cdot U(10)$	$\not\subset W$	self-dual
(14)	$S^{12}_{11,11}$	$(1+e_1^*, e_1e_2, e_2e_3+e_3^*)$	$(GL(1)^2\times SL(2))\cdot U(12)$	$-6s-18/2$	self-dual
(15)	$S^9_{11,15}$	$(1, e_1e_2, e_1e_3+e_1^*)$	$(GL(1)^2\times SL(2)\times SL(2))\cdot U(9)$	$\not\subset W$	$S^{13}_{15,11}$
(16)	$S^{11}_{13,13}$	$(1, e_1e_2, e_3e_4+e_3^*)$	$(GL(1)^2\times SL(2)\times SL(2))\cdot U(11)$	$\not\subset W$	$S^{14}_{13,13}$
(17)	$S^{14}_{13,13}$	$(1+e_1^*, e_2e_3+e_2^*, e_3e_4)$	$(GL(1)^2\times SL(2))\cdot U(14)$	$\not\subset W$	$S^{11}_{13,13}$
(18)	$S^7_{14,6}$	$(1, e_1e_2, e_1^*)$	$(GL(1)^2\times SL(2)\times SL(3))\cdot U(7)$	$-7s-20/2$	$S^9_{6,14}$
(19)	$S^2_{14,30}$	$(1+e_1^*, e_1e_2+e_2^*, 0)$	$(GL(1)\times G_2\times SL(2))\cdot G_a^2$	N.P.	$S^{10}_{30,14}$
(20)	$S^{15}_{15,5}$	$(1, e_1e_2, e_1e_3+e_3^*)$	$(GL(1)^3\times SL(2))\cdot U(15)$	$\not\subset W$	$S^7_{5,15}$
(21)	$S^{13}_{15,11}$	$(1+e_1^*, e_2e_3+e_2^*, 0)$	$(GL(1)^2\times SL(2)\times SL(2))\cdot U(13)$	$\not\subset W$	$S^9_{11,15}$
(22)	$S^8_{16,16}$	$(1, e_1e_2+e_3e_4, e_5^*)$	$(GL(1)\times SL(2)\times Sp(2))\cdot U(8)$	N.P.	self-dual
(23)	$S^{14}_{17,7}$	$(1, e_1e_2, e_3e_4)$	$(GL(1)^3\times SL(2)\times SL(2))\cdot U(14)$	N.P.	$S^{10}_{7,17}$
(24)	$S^{13}_{18,18}$	$(1, e_1e_2+e_1^*, 0)$	$(GL(1)^3\times SL(3))\cdot U(13)$	$\not\subset W$	self-dual
(25)	$S^{17}_{19,3}$	$(1, e_1e_2, e_1e_3+e_2e_4)$	$(GL(1)^2\times SL(2)\times SL(2))\cdot U(17)$	$-10s-35/2$	$S^5_{3,19}$
(26)	$S^{10}_{22,6}$	$(1, e_1^*, 0)$	$(GL(1)^3\times SL(4))\cdot U(10)$	$\not\subset W$	$S^2_{6,22}$
(27)	$S^{14}_{23,5}$	$(1, e_1e_2, e_1e_3)$	$(GL(1)^2\times SL(2)\times SL(3))\cdot U(16)$	N.P.	$S^7_{5,23}$
(28)	$S^{16}_{23,7}$	$(1, e_1e_2+e_3e_4, 0)$	$(GL(1)^3\times Sp(2))\cdot U(16)$	$-9s-32/2$	$S^8_{7,23}$
(29)	$S^{17}_{27,1}$	$(1, e_1e_2, 0)$	$(GL(1)^2\times SL(2)\times SL(2)\times SL(3))\cdot U(17)$	$-11s-41/2$	$S^7_{1,27}$
(30)	$S^{10}_{30,14}$	$(1+e_1^*, 0, 0)$	$(GL(1)^2\times SL(2)\times Spin(7))\cdot U(10)$	N.P.	$S^2_{14,30}$
(31)	$S^{12}_{35,3}$	$(1, 0, 0)$	$(GL(1)^2\times SL(2)\times SL(5))\cdot G_a^{12}$	$\not\subset W$	$S^4_{3,35}$
(32)	$S^0_{48,0}$	$(0, 0, 0)$	$Spin(10)\times GL(3)$	$-12s-48/2$	$S^0_{0,48}$

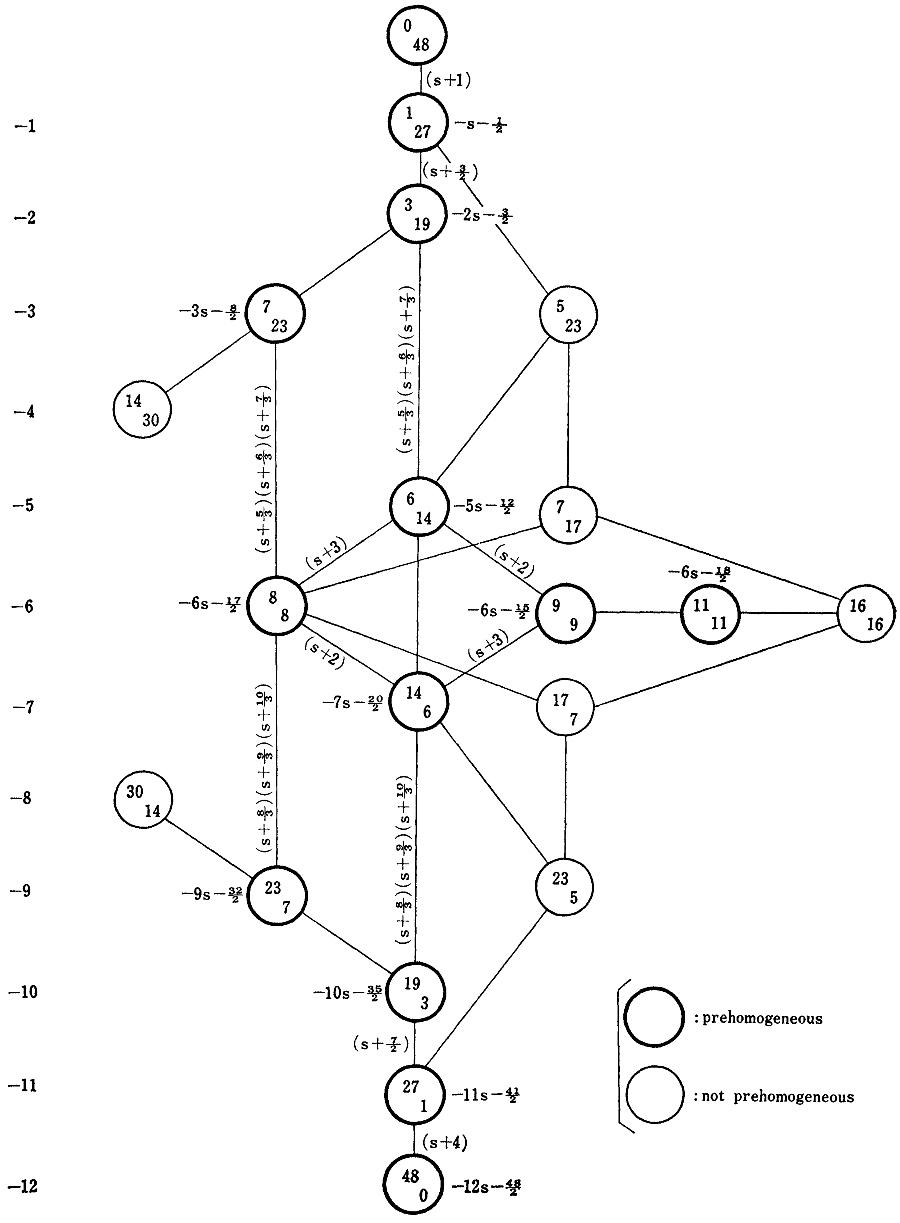

Fig. 1. The Holonomy Diagram of $(Spin(10)\times GL(3),\ \rho_1\otimes\Lambda_1,\ V(16)\otimes V(3))$.

§2. Holonomy diagram and the b-function. The holonomy diagram of (G, ρ, V) is given in Fig. 1, where $\textcircled{ij}$ stands for Λ_{ij}^{k}.

The intersections $\textcircled{0_{48}}-\textcircled{1_{27}}-\textcircled{3_{19}}$ and $\textcircled{9_{9}}-\textcircled{6_{14}}-\textcircled{8_{8}}-\textcircled{7_{23}}$

are all G_0-prehomogeneous, with $G_0 = Spin(10) \times SL(3)$. Since $\textcircled{0_{48}}$ is clearly in W, $\textcircled{1_{27}}$, $\textcircled{3_{19}}$ and their duals are contained in W (see Prop. 6.6 in [2]). To show that $\textcircled{9_{9}}$, $\textcircled{6_{14}}$, $\textcircled{8_{8}}$, $\textcircled{7_{23}}$ and their duals are in W, it is enough to prove the following lemma.

Lemma. $\Lambda_{8,8}^{11} \subset W$.

Proof. Put $x_8 = (1, e_1e_2 + e_3e_4, e_1e_3 + e_1^*)$ and $y_8 = (2e_5^* - (1/2)e_3e_5, (3/2)e_3e_4 - (1/2)e_4^*, -(3/2)e_1e_5)$. Then, (x_8, y_8) is a point of the Zariski-dense G-orbit in $\Lambda_{8,8}^{11}$. Now put $x = x_8 - (e_3e_5 + e_5^*, e_4e_5 + e_4^*, e_1e_5)$, $y = y_8 + (3/2, (3/2)e_1e_2 + (1/2)e_3e_4, (1/2)e_1e_3 + 2e_1^*)$. Then we have $f(x) \neq 0$ and $y = \operatorname{grad} \log f(x)$, namely, $(x, y) \in W$. For $\varepsilon \in \boldsymbol{C}^{\times}$, put $g_\varepsilon = \begin{pmatrix} h_\varepsilon & 0 \\ 0 & {}^t h_\varepsilon^{-1} \end{pmatrix} \times \begin{pmatrix} \varepsilon^4 & & \\ & \varepsilon^2 & \\ & & 1 \end{pmatrix} \in G_{x_8}$ where $h_\varepsilon = \operatorname{diag}(\varepsilon^4, \varepsilon^{-2}, 1, \varepsilon^2, \varepsilon^4)$. Then we have $(x_8, y_8) = \lim_{\varepsilon \to 0} (\rho(g_\varepsilon)x, \varepsilon^4 \rho^*(g_\varepsilon)y) \in W$ and hence $\Lambda_{8,8}^{11} \subset W$. Q.E.D.

From Fig. 1 and Theorem 7.5 in [2], we obtain the b-function.

Proposition. *The b-function $b(s)$ of the relative invariant $f(x)$ is given by*

$$b(s) = (s+1)\left(s+\frac{3}{2}\right)(s+2)\left(s+\frac{5}{3}\right)\left(s+\frac{6}{3}\right)\left(s+\frac{7}{3}\right)\left(s+\frac{8}{3}\right)\left(s+\frac{9}{3}\right)$$
$$\times\left(s+\frac{10}{3}\right)(s+3)\left(s+\frac{7}{2}\right)(s+4).$$

Remark. For the conormal bundles $\Lambda_{11,11}^{12}$ and Λ which are not G-prehomogeneous, it is not known whether they are in W or not.

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