58. Conformally Related Product Riemannian Manifolds with Einstein Parts

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Introduction. Let $(M, \bar{g})$ and (M^*, g^*) be product Riemannian manifolds of dimension $n \geq 3$. A conformal diffeomorphism f of M to M^* is characterized by the metric change

$$g^* = \rho^{-2}\bar{g}, \tag{0.1}$$

where ρ is a positive valued scalar field. In a previous paper [4], the present authors have proved the following

Theorem A. *Let both M and M^* be complete, connected and simply connected product Riemannian manifolds of dimension $n \geq 3$. If there is a global non-homothetic conformal diffeomorphism f of M onto M^*, then the underlying manifold of M and M^* is the product $N_1 \times N_0 \times N_2$ of three complete Riemannian manifolds N_1, N_0 and N_2 and the associated scalar field ρ with f depends on one part, say N_0, only. If the metric forms of N_1, N_0 and N_2 are denoted by ds_1^2, ds_0^2 and ds_2^2 respectively, then* (1) *M is the product $M_1 \times N_2$, where M_1 is an irreducible complete Riemannian manifold, and the metric form of M is written as*

$$\rho^2 ds_1^2 + ds_0^2 + ds_2^2 \tag{0.2}$$

on the underlying manifold $N_1 \times N_0 \times N_2$, and (2) *$M^*$ is the product $N_1 \times M_2^*$, where M_2^* is an irreducible complete Riemannian manifold, and the metric form of M^* is written as*

$$ds_1^2 + \rho^{-2}(ds_0^2 + ds_2^2) \tag{0.3}$$

on the same underlying manifold $N_1 \times N_0 \times N_2$.

Two-dimensional manifolds are regarded as Einstein ones. The purpose of the present paper is to prove the following

Theorem. *In addition to the assumptions of Theorem A, we suppose that both the irreducible parts M_1 of M and M_2^* of M^* are Einstein manifolds. If there is a global non-homothetic conformal diffeomorphism of M onto M^*, then each part N_α $(\alpha = 1, 0, 2)$ of the underlying manifold $N_1 \times N_0 \times N_2$ of M and M^* is of dimension one and the curvatures of the two-dimensional parts M_1 and M_2^* are not constants.*

As an immediate consequence of this Theorem, we can state

Corollary. *If, on the manifolds M and M^* stated in Theorem A, the Ricci tensors of the irreducible parts M_1 of M and M_2^* of M^* are* *parallel, then there exists no global non-homothetic conformal diffeomorphism of M onto M^*.*

This is a generalization of a theorem due to N. Tanaka [2] and T. Nagano [1], see also [5].

1. Let $(M, \bar{g})$ and (M^*, g^*) be product Riemannian manifolds of dimension $n \geq 3$ and f a conformal diffeomorphism of M to M^* characterized by (0.1). With respect to a local coordinate system (x^κ) of M, we shall denote the metric tensor $\bar{g}$ of M by components $\bar{g}_{\mu\lambda}$, the Christoffel symbols by $\{^{\kappa}_{\mu\lambda}\}$, the curvature tensor by $K_{\nu\mu\lambda}{}^{\kappa}$, the Ricci tensor by $K_{\mu\lambda}$ and the scalar curvature by κ, where κ is defined by $n(n-1)\kappa = K_{\mu\lambda}\bar{g}^{\mu\lambda}$ for $n \geq 2$ and $\kappa = 0$ for $n = 1$. Here and hereafter, Greek indices run on the range 1 to n. Quantities of M^* corresponding to those of M under f are indicated by asterisking. Then we have in particular the transformation formulas of the Ricci tensors

$$K^*_{\mu\lambda} = K_{\mu\lambda} + \rho^{-1}(n-2)\bar{\nabla}_\mu \rho_\lambda + \rho^{-1}\bar{g}_{\mu\lambda}\bar{\nabla}_\kappa \rho^\kappa - \rho^{-2}(n-1)\Phi \bar{g}_{\mu\lambda}, \tag{1.1}$$

where we have denoted the covariant differentiation with respect to $\bar{g}$ by $\bar{\nabla}$ and put $\rho_\lambda = \bar{\nabla}_\lambda \rho$, $\rho^\kappa = \bar{g}^{\lambda\kappa}\rho_\lambda$ and $\Phi = \rho_\kappa \rho^\kappa$.

Under the assumptions of Theorem A, the underlying manifold of M and M^* is the same product manifold $N_1 \times N_0 \times N_2$. Let each part N_α be of dimension n_α $(\alpha = 1, 0, 2)$, $n_1 + n_0 + n_2 = n$. There is a local coordinate system $(x^k) = (x^a, x^h, x^p)$ in M and M^* such that (x^a), (x^h) and (x^p) are local coordinate systems of N_1, N_0 and N_2 respectively. Latin indices run on the following ranges:

$$a, b, c, d = 1, 2, \cdots, n_1; \qquad h, i, j, k = n_1 + 1, \cdots, n_1 + n_0;$$
$$p, q, r, s = n_1 + n_0 + 1, \cdots, n;$$
$$A, B, C, D = 1, 2, \cdots, n_1, n_1 + 1, \cdots, n_1 + n_0;$$
$$P, Q, R, S = n_1 + 1, \cdots, n_1 + n_0, n_1 + n_0 + 1, \cdots, n.$$

In such a coordinate system, we denote the components of the metric tensors of N_1, N_0 and N_2 by g_{cb}, g_{ji} and g_{rq} and the Christoffel symbols by Γ^a_{cb}, Γ^h_{ji} and Γ^p_{rq} respectively. Then the metric forms (0.2) of M and (0.3) of M^* are expressed as

$$\bar{g}_{\mu\lambda}dx^\mu dx^\lambda = \rho^2 g_{cb}dx^c dx^b + g_{ji}dx^j dx^i + g_{rq}dx^r dx^q,$$
$$g^*_{\mu\lambda}dx^\mu dx^\lambda = g_{cb}dx^c dx^b + \rho^{-2}g_{ji}dx^j dx^i + \rho^{-2}g_{rq}dx^r dx^q$$

respectively. Since ρ is a function of N_0 only, the Christoffel symbol $\{^{\kappa}_{\mu\lambda}\}$ of M has non-trivial components

$$\begin{cases} \{^{a}_{cb}\} = \Gamma^a_{cb}, & \{^{a}_{ci}\} = \rho^{-1}\rho_i \delta^a_c, \quad \{^{h}_{cb}\} = -\rho\rho^h g_{cb}, \\ \{^{h}_{ji}\} = \Gamma^h_{ji}, & \{^{p}_{rq}\} = \Gamma^p_{rq}, \end{cases} \tag{1.2}$$

and Φ depends on x^h only. It follows from (1.2) that N_1 is totally umbilical and N_0 and N_2 are totally geodesic in M. The covariant differentiation $\bar{\nabla}_\mu \rho_\lambda$ on M has non-trivial components

$$\bar{\nabla}_c \rho_b = \rho\Phi g_{cb} \quad \text{or} \quad \bar{\nabla}_c \rho^a = \rho^{-1}\Phi\delta^a_c, \quad \bar{\nabla}_j \rho_i = \nabla_j \rho_i, \tag{1.3}$$

where ∇_j is the covariant differentiation with respect to g_{ji}.

The Ricci tensors of N_1, N_0 and N_2 will be denoted by R_{cb}, R_{ji} and R_{rq} respectively. Then the Ricci tensor $K_{\mu\lambda}$ of M has non-trivial components

(1.4) $$K_{cb}=R_{cb}-[\rho\nabla_h\rho^h+(n_1-1)\Phi]g_{cb},$$

(1.5) $$K_{ji}=R_{ji}-n_1\rho^{-1}\nabla_j\rho_i,$$

(1.6) $$K_{rq}=R_{rq}.$$

The Ricci tensor $K^*_{\mu\lambda}$ of M^* has non-trivial components

(1.7) $$K^*_{cb}=R_{cb},$$

(1.8) $$K^*_{ji}=R_{ji}+\rho^{-2}[\rho\nabla_h\rho^h-(n_0+n_2-1)\Phi]g_{ji}+(n_0+n_2-2)\rho^{-1}\nabla_j\rho_i,$$

(1.9) $$K^*_{rq}=R_{rq}+\rho^{-2}[\rho\nabla_h\rho^h-(n_0+n_2-1)\Phi]g_{rq}.$$

2. Let us prove Theorem. Components of quantities on the irreducible parts M_1 of M and M_2^* of M^* will be indicated by using indices $A, B, C, \cdots$ and $P, Q, R, \cdots$ respectively. If the parts M_1 and M_2^* are Einstein ones, then the Ricci tensors K_{CB} of M_1 and K^*_{RQ} of M_2^* are given by

(2.1) $$K_{CB}=(n_1+n_0-1)\kappa_1\bar{g}_{CB},$$

(2.2) $$K^*_{RQ}=(n_0+n_2-1)\kappa_2^* g^*_{RQ},$$

where κ_1 and κ_2^* are the scalar curvatures of M_1 and M_2^* respectively.

Since $\bar{g}_{cb}=\rho^2 g_{cb}$ and $\bar{g}_{ji}=g_{ji}$, it follows from (1.4), (1.5) and (2.1) that the Ricci tensors R_{cb} of N_1 and R_{ji} of N_0 are equal to

(2.3) $$R_{cb}=[(n_1-1)\Phi+(n_1+n_0-1)\rho^2\kappa_1+\rho\nabla_h\rho^h]g_{cb},$$

(2.4) $$R_{ji}=(n_1+n_0-1)\kappa_1 g_{ji}+n_1\rho^{-1}\nabla_j\rho_i$$

respectively. For $n_1=1$, the scalar function in the brackets of (2.3) is equal to zero. For $n_1\geq 2$, that is, $\dim M_1\geq 3$, the scalar curvature κ_1 of M_1 is a constant and hence the scalar function depends only on N_0. Therefore the scalar function is a constant independently of the dimension of N_1.

Since $g^*_{ji}=\rho^{-2}g_{ji}$ and $g^*_{qp}=\rho^{-2}g_{qp}$, the Ricci tensors R_{ji} of N_0 and R_{rq} of N_2 are written as

(2.5) $$R_{ji}=[(n_0+n_2-1)\rho^{-2}(\kappa_2^*+\Phi)-\rho^{-1}\nabla_h\rho^h]g_{ji}-(n_0+n_2-2)\rho^{-1}\nabla_j\rho_i,$$

(2.6) $$R_{rq}=[(n_0+n_2-1)\rho^{-2}(\kappa_2^*+\Phi)-\rho^{-1}\nabla_h\rho^h]g_{rq}$$

by virtue of (1.8), (1.9) and (2.2) respectively. We can also see that the scalar function in the brackets of (2.6) is a constant.

Comparing (2.4) with (2.5), we have the equation

(2.7) $$(n-2)\nabla_j\rho_i=-[(n_1+n_0-1)\kappa_1-(n_0+n_2-1)\rho^{-2}(\kappa_2^*+\Phi)+\rho^{-1}\nabla_h\rho^h]\rho g_{ji}.$$

If $\dim M_1>3$ or the scalar curvature κ_1 of M_1 for $\dim M_1=2$ is a constant, then coefficient in the brackets of (2.7) is equal to a constant, say $(n-2)k$. The equation (2.7) is then reduced to

(2.8) $$\nabla_j\rho_i=-k\rho g_{ji}.$$

Since N_0 is totally geodesic in M_1 and M is the product $M_1\times N_2$, any geodesic curve in N_0 is geodesic in M. Let Γ be a geodesic curve lying in N_0 and s the arc length of Γ. The ordinary derivatives with respect to s will be denoted by prime. The equation (2.8) is reduced to the ordinary differential one

$$\rho''(s) = -k\rho \tag{2.9}$$

along Γ. According to the signature of k, we put $k=0$, $k=-c^2$ or $k=c^2$, where c is a positive constant. By a suitable choice of s, the solution of (2.9) along Γ is given by one of the following:

$$\rho(s) = \begin{cases} (1)\ as+b \quad \text{for } k=0; \\ (2)\ a \exp cs,\ (3)\ a \sinh cs \quad \text{or} \quad (4)\ a \cosh cs \quad \text{for } k=-c^2; \\ (5)\ a \cos cs \quad \text{for } k=c^2, \end{cases}$$

where a and b are arbitrary constants. In the case (1), there is a geodesic curve Γ such that the solution along Γ is given by (1) with $a \neq 0$. Then $\rho(s) < 0$ on some interval of Γ. This contradicts to the completeness of N_0 and the fact that ρ is positive valued. We also see that the cases (3) and (5) do not occur.

Let Γ^* be the image $f(\Gamma)$ and s^* be the arc length of Γ^* such as $s^*=0$ corresponding to $s=0$. In the case (2), s and s^* are related by

$$ds^*/ds = \rho^{-1} = a^{-1} \exp(-cs),$$

or, by integration of this equation,

$$s^* = (1/ac)[1-\exp(-cs)] < 1/ac.$$

Therefore the arc length s^* of Γ^* is bounded as $s \to \infty$ along Γ. This is a contradiction (see [5]). In the case (4), s and s^* are related by

$$s^* = (2c/a)(\arctan \exp cs - \pi/4) < c\pi/2a,$$

and this leads a contradiction too. Thus, if there is a global non-homothetic conformal diffeomorphism of M onto M^*, then $\dim M_1 = 2$, that is, $n_1 = n_0 = 1$, and the curvature of M_1 is not constant. Similarly M_2^* should be a two-dimensional manifold with non-constant curvature.

References

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