56. *Rational Functions of (0, 1)-Type on the Two-Dimensional Complex Projective Space*

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1. Let R be a rational function on the two-dimensional complex projective space. Let $\{p_1, \cdots, p_r\}$ be the set of indeterminate points of R. If the irreducible components of a generic level curve of R in $\boldsymbol{P}^2-\{p_1, \cdots, p_r\}$ are an open Riemann surface of genus g with n points as its boundary, then R is called (g, n)-type. The purpose of this note is to give the explicit forms of all the rational functions of (0, 1)-type. The details will be published elsewhere. A rational function is called primitive if a generic level curve is irreducible. As any rational function is a composite of a primitive function by a rational function of one variable, we assume hereafter that R is primitive rational function of (0, 1)-type. Such a rational function R has only one indeterminate point and all the level curves are irreducible. There are at most two level curves with order larger than one which we shall call singular level curves. Here a level curve $R^{-1}(\alpha)$ $(\alpha \in \boldsymbol{P}^1)$ has order m if $R-\alpha$ or $1/R$ takes zero with order m. Thus the set $\mathcal{F}$ of primitive functions of (0, 1)-type decomposes into the three parts $\mathcal{F}_0$, $\mathcal{F}_{\mathrm{I}}$ and $\mathcal{F}_{\mathrm{II}}$ according to the number of singular level curves. The set $\mathcal{F}_0$ consists of linear functions on $\boldsymbol{P}^2$. The explicit forms of rational functions in $\mathcal{F}_{\mathrm{I}}$ is rather simple because the singular level curve is a line. We omit this case in this note.

2. Let R be a primitive rational function in $\mathcal{F}_{\mathrm{II}}$. Then we can assume without loss of generality that its divisor has the form

$$(R)=mS_0-nS_\infty \tag{1}$$

where m and n are integers relatively prime and $1<m<n$. We can take a rational function φ on $\boldsymbol{P}^2$ such that (R, φ) is an isomorphism from $\boldsymbol{P}^2-(S_0 \cup S_\infty)$ onto $\boldsymbol{C}^* \times \boldsymbol{C}$. Furthermore, φ takes zero on S_0 of order less than m and has pole on S_∞ of order less than n. By these conditions φ is uniquely determined up to a constant multiple. This function is of (0, 2)-type and its divisor has the form

$$(\varphi)=T+sS_0-tS_\infty \tag{2}$$

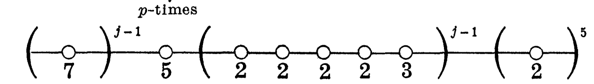
where T is an irreducible curve and s and t are integers satisfying $1 \leqq s<m$, $s<t<n$ and $(m, s)=1$. The degrees of the curves S_0, S_∞ and T are n, m and $mt-ns$, respectively. Then, the rational function

$f=\varphi^m/R^s$ is of (0, 2)-type, and f takes a constant value on S_0. We normalize this value to be -1. In this case we call $[R, \varphi]$ a normalized pair. Then we have

(3) $$(f+1)=S_0+U-(mt-ns)S_\infty$$

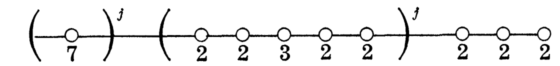
for an irreducible curve U.

3. By successive blowing-up's at the indeterminate point of R, we obtain $\rho: M\to P^2$ so that $R\cdot\rho$ has no more indeterminate point. Let Σ be the inverse image of the indeterminate point of R by ρ. Then we can explicitely give the graph of Σ. Here the graph of Σ means the set of dots, which represents the irreducible components of Σ. If two irreducible components intersect, we join the two corresponding dots by the segment. The integer attached to the dot gives the absolute value a of the self-intersection number $-a$ of the corresponding component. In order to explain the possible graph for Σ, we prepare several notations. For a graph G, we mean by G^p the graph $\underbrace{G-G-\cdots-G}_{p\text{-times}}$. For $l\geqq 0$, by $\vec{G}_l$ we denote the graph



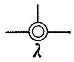
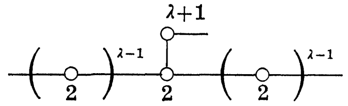
for $l=2j-1$ $(j\geqq 1)$

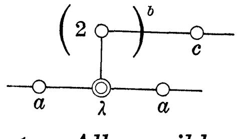
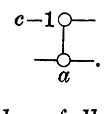
and



for $l=2j$ $(j\geqq 0)$.

We understand $\vec{G}_{-1}$ by the empty graph. By $\vec{H}_l$ $(l\geqq 1)$, we denote the graph deleted five $\underset{2}{\circ}$ in the right end from $\vec{G}_l$. By ${}^+\vec{G}_l$ (resp. $\vec{G}_l^+$) $(l\geqq 0)$, we denote the graph obtained from $\vec{G}_l$ by increasing the number on the extremely left (resp. right) by one. ${}^+\vec{H}_l$ is obtained from $\vec{H}_l$ by increasing the number of the extreme left by one and ${}^+\vec{H}^-$ is obtained from ${}^+\vec{H}_l$ by decreasing the extreme right number by one. The graphs $\overleftarrow{G}_l$, $\overleftarrow{H}_l$, etc. are the graphs obtained by reversing direction.

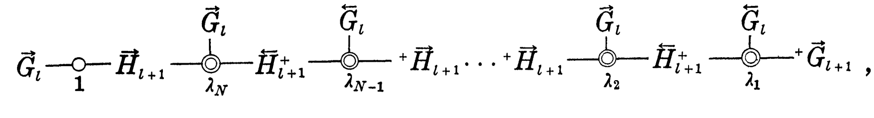
The graph  means  when $\lambda\geqq 1$.

When $\lambda=0$,  $(b\geqq 0,\ c\geqq 3)$ means .

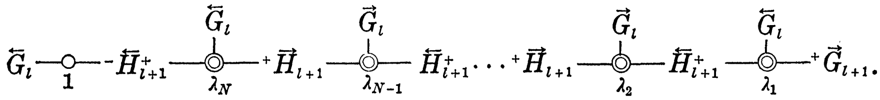
Theorem 1. *All possible graphs for Σ are listed as follows.*

II(l) $(l\geqq 0)$: $\overleftarrow{G}_l \underset{1}{-\circ-} \vec{G}_{l+1}$.

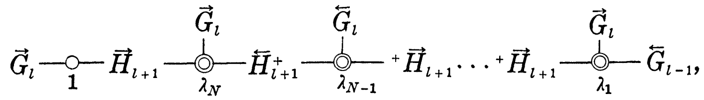
II$^+(l, N; \lambda_1, \cdots, \lambda_N)$: *when N is a positive even integer* $(N\geqq 2)$



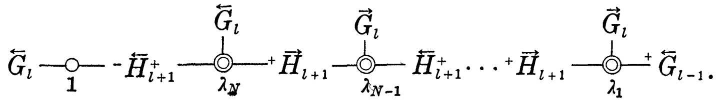
when N is an odd integer ($N \geqq 1$)



$\mathrm{II}^-(l, N; \lambda_1, \cdots, \lambda_N)$: *when N is an odd integer* ($N \geqq 1$)

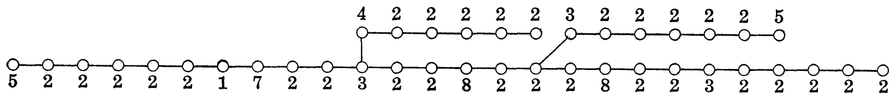


when N is an even integer ($N \geqq 2$)



Here $\lambda_1, \cdots, \lambda_N$ are integers such that $\lambda_j \geqq 0$ when $N \geqq 1$ and $\lambda_j \geqq 1$ when $N=0$.

For example, $\mathrm{II}^+(1, 2; 2, 0)$ means the graph



4. Let $[R, \varphi]$ be a normalized pair belonging to $\mathrm{II}(l)$. Then, m, n, s, t and $mt-ns$ for the $[R, \varphi]$ given by (1), (2) and (3) are respectively m_l, m_{l+1}, s_l, s_{l+1} and 3 defined by $m_l = 3m_{l-1} - m_{l-2}$, $m_0 = 2$, $m_1 = 5$ and $s_l = m_l - m_{l-2}$. For $[R, \varphi]$ belonging to $\mathrm{II}^{\pm}(l, N, \lambda_1, \cdots, \lambda_N)$, m, n, s, and t are respectively m_l, $n_l^{\pm}(N, \lambda_1, \cdots, \lambda_N)$, $s_l^{\pm}(N)$ and $t_l^{\pm}(N, \lambda_1, \cdots, \lambda_N)$ given as follows; $s_l^+(N) = s_l$ for an even N, $= m_{l-2}$ for an odd N, $s_l^-(N) = s_l^+(N-1)$, $n_l^{\pm}(N, \lambda_1, \cdots, \lambda_N) = m_{l\pm 1} \prod_{i=1}^{N} (\lambda_i m_l^2 + m_l s_l^{\pm}(i) - 1)$ and $t_l^{\pm}(N, \lambda_1, \cdots, \lambda_N) = (\lambda_N m_l s_l^{\pm}(N) + (s_l^{\pm}(N))^2 + \lambda_N) n_l^{\pm}(N-1, \lambda_1, \cdots, \lambda_{N-1})$, where $n_l^{\pm}(0) = m_{l+1}$.

Theorem 2. *We define the rational functions R_l and φ_l by*

$$R_l = (\varphi_{l-1}^{m_l} + R_{l-1}^{s_l})^{m_l} / (R_{l-1})^{-1+s_l m_l}, \quad \varphi_l = \varphi_{l-1} (\varphi_{l-1}^{m_l} + R_{l-1}^{s_l})^{s_l} / (R_{l-1})^{(s_l)^2}$$

for $l \geqq 1$ and

$$R_0 = \{(y-x^2)^2 - 2xy^2(y-x^2) + y^5\}^2 / (y-x^2)^5$$
$$\varphi_0 = (xy - x^3 - y^3)\{(y-x^2)^2 - 2xy^2(y-x^2) + y^5\} / (y-x^2)^4$$

for an inhomogeneous coordinates (x, y) of $\boldsymbol{P}^2$. Then (R_l, φ_l) is a normalized pair in $\mathrm{II}(l)$. *Conversely, any $[R, \varphi]$ in* $\mathrm{II}(l)$ *has the form $[c^{m_l} R_l, c^{s_l} \varphi_l]$ for some $c \in \boldsymbol{C}^*$ and some inhomogeneous coordinates (x, y) of $\boldsymbol{P}^2$.*

In this case, S_∞, U and T for $[R_l, \varphi_l]$ given by (1), (2) and (3) concide with S_0, S_∞ and T for $[R_{l-1}, \varphi_{l-1}]$ respectively.

Theorem 3. *If (R, φ) is a normalized pair belonging to $\mathrm{II}^{\pm}(l, N; \lambda_1, \cdots, \lambda_N)$, then there exist unique $a_j \in \boldsymbol{C}$ ($j = 1, \cdots, \lambda_N$),*

$a_{1+\lambda_N} \in \boldsymbol{C}^*$ *and a normalized pair* $[R', \varphi']$ *belong to* $\mathrm{II}^{\pm}(l, N-1, \lambda_1, \cdots, \lambda_{N-1})$ *such that*

$$R = P^{m_l}/(R')^{1+m_l s_l^{\pm}(N-1)}, \qquad \varphi = (\xi P^{s_l^{\pm}(N)})/(R')^{1+m_l-2s_l}$$

where $\xi = \varphi' + \sum_{j=1}^{1+\lambda_N} a_j (R')^j$ *and* $P = \xi^{m_l} + (R')^{s_l^{\pm}(N-1)}$. *Conversely the* (R, φ) *given in this way is a normalized pair in* $\mathrm{II}^{\pm}(l, N, \lambda_1, \cdots, \lambda_N)$. *Here we assume* $l \geqq 0$, $N \geqq 1$ *for* II^+ *and* $l \geqq 0$, $N \geqq 2$ *for* II^-.

In this case, U and S_∞ for $[R, \varphi]$ are S_0 and S_∞ for $[R', \varphi']$ respectively.

Theorem 4. *For* $a_j \in \boldsymbol{C}$ $(j = 0, \cdots, \lambda - 1)$ *and* $a_\lambda \in \boldsymbol{C}^*$, *we set*

$$R = P^{m_l}/(R_{l-1})^{-1+m_l s_l}, \qquad \varphi = (\xi P^{s_l})/(R_{l-1})^{(s_l)^2}$$

when $\xi = \varphi_{l-1} + \sum_{j=0}^{\lambda} a_j R_{l-1}^{-j}$ *and* $P = \xi^{m_l} + R_{l-1}^{s_l}$. *Then* (R, φ) *is a normalized pair in* $\mathrm{II}^-(l, 1; \lambda)$. *When* $l = 0$, *we assume* $a_0 = 0$ *and we set*

$$R_{-1} = y^{-2}(y - x^2) \quad \text{and} \quad \varphi_{-1} = y^{-1}x.$$

Conversely any $[R, \varphi]$ *in* $\mathrm{II}^-(l, 1; \lambda)$ *is written in this form up to a constant multiple.*

In this case, S_∞ and U for $[R, \varphi]$ are S_0 and S_∞ for $[R_{l-1}, \varphi_{l-1}]$ respectively.

As corollary to the above results, we can give explicitly the equations defining S_0, S_∞ and T by the similar recursion formula.

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