54. A Complement to Trotter's Product Formula for Nonlinear Semigroups Generated by the Subdifferentials of Convex Functionals*)

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The purpose of this note is to announce several new results concerning product formulas for nonlinear semigroups in Hilbert space. Our main result (Theorem 4) complements the corresponding results in [4] and [5] because convergence is obtained for all points in the space. It also leads to a negative solution of a problem of H. Brezis [2, Problem 24].

We begin with two general theorems which are valid in any Banach space. We denote the closure of a set B by cl (B).

Theorem 1. *Let E be a Banach space, D a closed subset of E, and $F(t): D \to D$, $0<t<\infty$, a family of nonexpansive mappings such that $R(I+r(I-F(t))) \supset D$ for all $t>0$ and $r>0$. If*

(1) $\lim_{p\to 0+} (I+(r/p)(I-F(p)))^{-1}x = J_r x$ *exists for each $x \in D$ and $r>0$, and $C = \mathrm{cl}\,\{J_r x : x \in D \text{ and } r>0\}$,*

then

(2) $\lim_{n\to\infty} F(t/n)^n x = S(t)x$ *exists for each $x \in C$ and $t \geq 0$, uniformly on compact t intervals.*

It can be shown that if (1) holds, then $\lim_{p\to 0+} F(p)y = y$ for each y in C. Therefore we let $F(0)$ be the identity in (2). Theorem 1 improves upon [7, Theorem 2] because $F(t)$ is not assumed to map D into C. The limit J_r is the resolvent (on D) of an accretive operator A that satisfies the range condition, and S is the semigroup generated by $-A$ on C.

Theorem 2. *Let E be a Banach space, D a closed subset of E, and $F(t): D \to D$, $0 \leq t < \infty$, a continuous family of nonexpansive mappings. Let C denote the non-empty fixed point set of $F(0)$ and assume that (2) holds. If*

(3) $\lim_{n\to\infty} F(0)^n x = Qx$ *exists for each $x \in D$,*

then

(4) $\lim_{n\to\infty} F(t/n)^n x = S(t)Qx$ *exists for each $x \in D$ and $t \geq 0$, uniform-*

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ly on compact t intervals.

Our next result follows from several Banach space theorems in [3] and [8].

Theorem 3. *Let H be a Hilbert space and let $P_k: H \to C_k$, $1 \le k \le m$, be the nearest point projections onto closed convex subsets $\{C_k : 1 \le k \le m\}$ of H. Assume that the set $\cap\{C_k : 1 \le k \le m\}$ is not empty, and let $0 < a_k < 1$ satisfy $\sum_{k=1}^{m} a_k = 1$. Then the weak $\lim_{n\to\infty} (P_m P_{m-1} \cdots P_1)^n x = Qx$ and $\lim_{n\to\infty} \left(\sum_{k=1}^{m} a_k P_k\right)^n x = Rx$ exist for all x in H and define nonexpansive retractions onto $\cap\{C_k : 1 \le k \le m\}$. If each C_k is symmetric with respect to the origin, then the convergence is strong.*

Let H be a Hilbert space, and let $\phi_k : H \to (-\infty, \infty]$, $1 \le k \le m$, be proper lower semicontinuous convex functions. Let A_k denote the subdifferential $\partial\phi_k$ of ϕ_k, S_k the semigroup generated by $-A_k$ on $C_k = \mathrm{cl}\,(D(A_k))$, $R_k(t)$ the resolvent $(I + tA_k)^{-1}$ of A_k, and $P_k : H \to C_k$ the nearest point projection onto C_k. Let A be the subdifferential of $\phi = \sum_{k=1}^{m} \phi_k$ (which is assumed to be proper), and S the semigroup generated by $-A$ on $C = \mathrm{cl}\,(D(A))$. If $0 < a_k < 1$ and $\sum_{k=1}^{m} a_k = 1$, we also define $\psi : H \to (-\infty, \infty]$ by $\psi = \sum_{k=1}^{m} a_k \phi_k$, let B be the subdifferential of ψ, and let T denote the semigroup generated by $-B$ on C. Our main result is now obtained by combining Theorems 1, 2 and 3 with the results of [4] and [5]. It improves upon Corollary 1.2 of [4] and Corollaries 12 and 13 of [5] because convergence is seen to hold for all x in H.

Theorem 4. *Assume either that each ϕ_k, $1 \le k \le m$, is even, or that H is finite-dimensional. Assume further that $C = \cap\{C_k : 1 \le k \le m\}$. Then the strong $\lim_{n\to\infty} (P_m P_{m-1} \cdots P_1)^n x = Qx$ and $\lim_{n\to\infty} \left(\sum_{k=1}^{m} a_k P_k\right)^n x = Rx$ exist for all x in H, and we have*

(a) $$\lim_{n\to\infty} [S_m(t/n)P_m \cdots S_1(t/n)P_1]^n x = \lim_{n\to\infty} [R_m(t/n) \cdots R_1(t/n)]^n x = S(t)Qx,$$

and

(b) $$\lim_{n\to\infty} \left[\sum_{k=1}^{m} a_k S_k(t/n) P_k\right]^n x = \lim_{n\to\infty} \left[\sum_{k=1}^{m} a_k R_k(t/n)\right]^n x = T(t)Rx$$

for all x in H and $t \ge 0$, uniformly on compact t intervals.

Let $m = 2$, for example, and assume that $A_1 + A_2$ is maximal monotone. Then $A_1 + A_2 = A$ and Theorem 4 is applicable whenever

$\mathrm{cl}\,(D(A_1)\cap D(A_2))=\mathrm{cl}\,(D(A_1))\cap \mathrm{cl}\,(D(A_2))$. For sufficient conditions that guarantee this and several concrete examples see [1]. Theorem 4 also leads to a negative solution of a problem of H. Brezis [2, Problem 24]. More details, as well as a negative solution to Problem 23 there, can be found in [9].

We conclude with a general result on the equivalence between resolvent consistency (1) and convergence (2). It improves upon Theorem 2.1 of [6] because $F(t)$ is no longer assumed to map D into C. It provides a partial answer to the query on p. 385 of [5] and is new even in Hilbert space.

Theorem 5. *Let D be a closed convex subset of a uniformly smooth Banach space. Let $F(t): D\to D$, $0\le t<\infty$, be a continuous family of nonexpansive mappings and let C denote the nonempty fixed point set of $F(0)$. Then* (1) *and* (2) *are equivalent.*

It is expected that detailed proofs and discussions of the results announced here, as well as other related results, will appear elsewhere.

References

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