50. *Euler's Finite Difference Scheme and Chaos in* $\mathbf{R}^n$

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1. **Introduction.** Let $F: R^n \to R^n$ be a continuously differentiable mapping. Consider an n-dimensional autonomous differential equation of the form

$$\frac{du}{dt} = F(u), \qquad u \in R^n. \tag{1}$$

Suppose that (1) has at least two equilibrium points $\bar{u}$ and $\bar{v}$. Under some conditions it is shown that the corresponding difference equations (Euler's scheme) for (1) are chaotic in the sense of Li and Yorke [1]. The theorem of Marotto [2] will be used to prove the existence of chaos. More precisely, it will be shown that both $\bar{u}$ and $\bar{v}$ are snap-back repellers.

This work is motivated by a theorem proven by Yamaguti and Matano [3] concerning scalar differential equations. I would like to thank Prof. M. Yamaguti for his interest and encouragement.

2. **Notation and theorem.** Euler's difference scheme for (1) takes the form

$$\begin{cases} u_1^{k+1} = u_1^k + \Delta t F_1(u_1^k, u_2^k, \cdots, u_n^k) \\ \vdots \qquad \vdots \\ u_n^{k+1} = u_n^k + \Delta t F_n(u_1^k, u_2^k, \cdots, u_n^k) \end{cases} \tag{2}$$

that is,

$$u^{k+1} = u^k + \Delta t F(u^k).$$

Letting $\Delta t = s$ and $G_s = Id + sF$, (2) implies

$$u^{k+1} = G_s(u^k). \tag{3}$$

For differentiable function f, let $f'(x)$ denote the Jacobian matrix of f at $x \in R^n$ and $\det f'(x)$ its determinant. Note that $G_s'(x) = E + sF'(x)$ for all $x \in R^n$ where E is a unit matrix. Let $B(x, r)$ denote the closed ball in R^n of radius r centered at x and $\|x\|$ be the usual Euclidean norm of x in R^n. For a square matrix A, let A^* denote the adjoint matrix of A. Our theorem can now be stated as follows:

Theorem. *Let F be continuous differentiable in R^n. Suppose there exist $\bar{u} \neq \bar{v}$ such that $F(\bar{u}) = F(\bar{v}) = 0$, $\det F'(\bar{u}) \neq 0$ and $\det F'(\bar{v}) \neq 0$. Then there exists a positive constant c such that for any $s > c$ the difference equation* (3) *is chaotic in the sense of Li and Yorke.*

Remark. The condition in the above theorem is a stable property under small perturbations of F.

3. Proof of the theorem. Before proving theorem we shall present three preliminary lemmas.

Lemma 1. *There exist* $r_1>0$ *and* $c_1>0$ *such that* $\det G_s'(x)\neq 0$ *for any* $s>c_1$ *and any* $x\in B(\bar{u},r_1)\cup B(\bar{v},r_1)$.

Proof. From the assumption on F, we can find $r_1>0$ such that $\det F'(x)\neq 0$ for any $x\in B(\bar{u},r_1)\cup B(\bar{v},r_1)$. If the lemma is false, then there exist two sequences $s_n>0$ and $x_n\in B(\bar{u},r_1)\cup B(\bar{v},r_1)$ such that $s_n\to\infty$ as $n\to\infty$ and $\det G_{s_n}'(x_n)=0$. Since $G_s'=E+sF'$, we have $\det[E/s_n+F'(x_n)]=0$. Without loss of generality we can assume that $x_n\to x^*\in B(\bar{u},r_1)\cup B(\bar{v},r_1)$ as $n\to\infty$. Thus, letting n tend to infinity, we find that $\det F'(x^*)=0$. This contradiction completes the proof.

Lemma 2. *For any* $\delta>1$, *there exist* $r_2>0$ *and* $c_2(\delta)>0$ *such that* $\|G_s(x)-G_s(y)\|\geq\delta\|x-y\|$ *for any* $s>c_2(\delta)$ *and any* $x,y\in B(\bar{u},r_2)$.

Proof. Since $\det F'(\bar{u})^*F'(\bar{u})=(\det F'(\bar{u}))^2\neq 0$, the least eigenvalue $\lambda_{\min}$ of a positive-semidefinite symmetric matrix $F'(\bar{u})^*F'(\bar{u})$ is positive. Hence

$$\|F'(\bar{u})x\|\geq\sqrt{\lambda_{\min}}\|x\| \qquad \text{for all } x\in R^n,$$

and there exists $r_2>0$ such that

$$\|F'(x)-F'(\bar{u})\|<\frac{1}{2}\sqrt{\lambda_{\min}} \qquad \text{for any } x\in B(\bar{u},r_2).$$

Therefore

$$\begin{aligned}\|F(x)-F(y)\|&=\left\|\int_0^1 F'(y+t(x-y))(x-y)dt\right\|\\ &\geq\|F'(\bar{u})(x-y)\|-\frac{1}{2}\sqrt{\lambda_{\min}}\|x-y\|\\ &\geq\frac{1}{2}\sqrt{\lambda_{\min}}\|x-y\| \qquad \text{for any } x,y\in B(\bar{u},r_2).\end{aligned}$$

Hence

$$\begin{aligned}\|G_s(x)-G_s(y)\|&\geq s\|F(x)-F(y)\|-\|x-y\|\\ &\geq\left(\frac{s}{2}\sqrt{\lambda_{\min}}-1\right)\|x-y\|\geq\delta\|x-y\|\end{aligned}$$

where $s>c_2(\delta)\equiv\dfrac{2}{\sqrt{\lambda_{\min}}}(1+\delta)$.

Lemma 3. *For a sufficiently small open neighbourhood* U *of* $\bar{u}$ *and any bounded set* W, *there exists* $c_3(U,W)>0$ *such that the equation* $G_s(u)=w$ *has at least one solution* $u\in U$ *for any* $s>c_3(U,W)$ *and any* $w\in W$.

Proof. Without loss of generality we can assume that $F|_{\bar{U}}$ is a homeomorphism. Since $\bar{u}$ is an isolated zero in $\bar{U}$, we have

$$\deg(0,F,U)=\text{sign}\det F'(\bar{u})=1 \text{ or } -1.$$

Now assume that $\mu_0u_0+F(u_0)=\mu_0w_0$ for some $u_0\in\partial U$, $w_0\in W$ and $\mu_0>0$. Then

$$\mu_0=\frac{\|F(u_0)\|}{\|u_0-w_0\|}\geq\frac{\inf_{u\in\partial U}\|F(u)\|}{\sup_{u\in\partial U,w\in W}\|u-w\|}\equiv\mu(U,W)>0.$$

Hence we obtain $\mu w \notin (\mu Id+F)(\partial U)$ for any $w\in W$ and any $0\leq\mu<\mu(U,W)$.

Consider now the homotopy $\nu x+F(x)$, $(\nu,x)\in[0,\mu]\times\overline{U}$ and by the homotopy property of mapping degree we have

$$\deg(\mu w,\mu Id+F,U)=\deg(0,F,U)\neq 0.$$

Therefore there exists $u\in U$ such that

$$(\mu Id+F)(u)=\mu w,\text{ that is, } G_{1/\mu}(u)=w.$$

This completes the proof.

Note that the similar arguments hold for $\bar{v}$. Now we are ready for the proof of the theorem. Select sufficiently small open neighbourhoods U,V of $\bar{u},\bar{v}$ respectively such that $U\cap V=\phi$ and Lemma 3 holds for both $\bar{u}$ and $\bar{v}$. Let $r^*=\min(r_1,r_2)$ and $c^*=\max(c_1,c_2(\delta),c_3(U,V),c_3(V,U))$. Without loss of generality we can assume that

$$U\subset B(\bar{u},r^*)\quad\text{and}\quad V\subset B(\bar{v},r^*).$$

By Lemma 3, for any $s>c^*$, there exist $v_s\in V$ and $u_s\in U$ such that $G_s(v_s)=\bar{u}$ and $G_s(u_s)=v_s$. Since $\det G_s'(u_s)\neq 0$ and $\det G_s'(v_s)\neq 0$ by Lemma 1, we can find $r_s>0$ such that $B(u_s,r_s)\subset B(\bar{u},r^*)$, $G_s(B(u_s,r_s))\subset V$, $G_s^2(B(u_s,r_s))\subset U$, and both $G_s|_{B(u_s,r_s)}$ and $G_s|_{G_s(B(u_s,r_s))}$ are homeomorphisms. Finally define compact sets $\{B_k\}_{-\infty<k\leq 2}$ as follows:

$$B_1=G_s(B(u_s,r_s)),\quad B_2=G_s^2(B(u_s,r_s))\quad\text{and}$$
$$B_{-k}=G_s^{-k}(B(u_s,r_s))\qquad\text{for } k\geq 0,$$

since G_s^{-k} is well-defined by Lemma 2. This shows that $\bar{u}$ is a snap-back repeller. Obviously same argument holds for $\bar{v}$.

4. Application. We shall attempt to apply our theorem to quadratic differential systems of the form

$$(4)\qquad \frac{d}{dt}\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}=F(u_1,\cdots,u_n)=\begin{pmatrix}(a_1-b_{11}u_1-\cdots-b_{1n}u_n)u_1\\ \vdots\\ (a_n-b_{n1}u_1-\cdots-b_{nn}u_n)u_n\end{pmatrix}.$$

These systems include prey-predator and competition models which are discussed in [4]. Let $A=(a_1,\cdots,a_n)$, $B=(b_{ij})$ and $O=(0,\cdots,0)$. If $A\neq O$ and $\det B\neq 0$, then one can easily show that (4) has at least two equilibrium points O and $B^{-1}A$. Moreover $\det F'(O)=a_1\cdots a_n$ and $\det F'(B^{-1}A)=(-1)^n\bar{u}_1\cdots\bar{u}_n\det B$ where $B^{-1}A=(\bar{u}_1,\cdots,\bar{u}_n)$. Therefore the conclusion of the theorem holds for (4) if $a_1\cdots a_n\bar{u}_1\cdots\bar{u}_n\neq 0$ and $\det B\neq 0$.

Finally note that the condition that F has at least two zeros can be weakened in some cases.

For example,

$$(5)\qquad dx/dt=1-e^x,\qquad x\in R.$$

This scalar differential equation has a unique equilibrium point 0 which is asymptotically stable. However one can easily show that there exists a positive constant c_0 such that for each $\Delta t > c_0$ Euler's scheme for (5) is chaotic in some invarient interval.

References

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