

5. On Preimage and Range Sets of Meromorphic Functions^{*)}

By Fred GROSS^{**)} and Chung-Chun YANG^{***)}

(Communicated by Kôzaku YOSIDA, M. J. A., Jan. 12, 1982)

1. Introduction. We say that two meromorphic functions f and g share a value c provided that $f(z)=c \iff g(z)=c$ (regardless of multiplicities). In this paper following Gundersen [3] we shall use the abbreviation CM=counting multiplicities and IM=ignoring multiplicities. In [3] it is shown that if two meromorphic functions f and g share three distinct values CM and share a fourth value IM, then they share all four values CM. Several growth relationships between f and g were also established in [3] when they share 3 or 4 values. Actually going back to 1929 the founder of value distribution theory R. Nevanlinna [6] proved that if two nonconstant meromorphic functions f and g share 5 distinct values (possibly including ∞) IM, then f and g must be identical. Thus, the study of the relationship between two meromorphic functions via the preimage sets of several distinct values in the range has a long history. However, only recently, have the studies been extended to include several preimage sets of several disjoint sets of (range) values. The first author of the present paper has already made some contributions on this aspect in [2]. In this paper we shall continue the study and provide some answers to some of the open questions raised in [2], and more importantly we hope that the present results will stimulate additional research and interest in this area.

1.1 Definitions and notations. It is well-known that given any complex number c , every countable discrete set S is the preimage set of c under a certain meromorphic function f . To avoid this trivial case we define a set S to be a nontrivial preimage set (NPS) if S is a countable discrete set and there exists at least one nonlinear entire function f and a finite (range) set T of distinct values with $|T| \geq 2$ (where $|T|$ denotes the cardinality of T) such that $f^{-1}(T)=S$. Note that the elements in S need not be distinct. It is natural to ask: does there exist a discrete set S which is not an NPS at all? This can be answered in the affirmative. One can exhibit such sets explicitly according to an argument used in [8]. A generalization of this result

^{*)} The problems discussed in this paper were posed by the first author at the conference on function theory in Lexington, Kentucky, 1976.

^{**)} University of Maryland at Baltimore County, Baltimore, MD., U.S.A.

^{***)} Naval Research Laboratory, Washington, DC.

was stated in [2]. Actually given any finite set S , one can add an infinite discrete set S_1 such that $S \cap S_1 = \emptyset$ and $S \cup S_1$ is not an NPS. Obviously if S is an NPS for some entire function f then S is also an NPS for any function of the form of $af + b$, where a and b are constants and $a \neq 0$. Thus we call S a unique NPS (or UNPS) if S is an NPS and any two entire functions f and g having S as their NPS are linearly dependent i.e., $g = af + b$ for some constants a, b . We now introduce some analogous definitions for the range sets. T is said to be a unique range set (URS) if T is a discrete set such that if any two entire functions f and g satisfy $f^{-1}(T) = g^{-1}(T)$ CM, then $f \equiv g$.

1.2 Infinite NPS and URS. Theorem 1. *Given any discrete set $\{a_n\}$ there exists a discrete set $\{b_n\}$ such that $S = \{a_n\} \cup \{b_n\}$ is not an NPS.*

Proof. Choose $\{b_n\}$ such that $b_1 = b_2 = b_3, b_4 = b_5 = b_6, \dots, b_{3n+1} = b_{3n+2} = b_{3n+3}, \dots$ with $b_1 \neq b_4 \neq b_7 \dots \neq b_{3n+1} \neq b_{3n+4}, \dots$ such that

$$\overline{\lim}_{r \rightarrow \infty} N(r, \{a_n\}) / N(r, \{b_n\}) = 0,$$

where $N(r, \{c_n\}) = \int_0^r \{n(t, \{c_n\})/t\} dt$ and $n(t, \{c_n\})$ is the number of points of $\{c_n\}$ satisfying $|c_n| \leq t$. It is readily seen that if the set $S = \{a_n\} \cup \{b_n\}$ is an NPS for some entire function f , then f would have at least two distinct complete ramified values of multiplicity equal to 3. This is impossible according to the Nevanlinna's theory on ramified values (more precisely, see [4, Theorem 2.4]).

Remark. This also answers question 8 raised in [2].

There do exist two distinct functions f and g having the same NPS. For instance, $S = \{0, \pm n\pi/2, n=1, 2, \dots\}$ is an NPS for $\sin z$ and $\cos z$ with the same range set $T = \{0, -1, 1\}$. The problem of finding a UNPS is more difficult and seems to require some extension of the classical Nevanlinna ramification theorem to algebroid functions.

Theorem 2. *Let f be an entire function of finite nonintegral order ρ with $\rho > 1$. Assume that all the zeros of f are real and simple. Let*

$$S = \{\xi \mid f^3(\xi) - 1 = 0\}.$$

Then S is a UNPS.

Proof. Clearly S is an NPS of f . Let g be another entire function which has S as its NPS. Then accordingly, we have

$$(1) \quad f^3 - 1 = p(g)e^{q(z)}$$

where $p(z)$ is a polynomial with 2 or more distinct roots, $q(z)$ is a polynomial with degree less than ρ . It follows from (1) that

$$T(r, g) = O(1)T(r, f)$$

and

$$T(r, e^q) = o(1)T(r, g)$$

as $r \rightarrow \infty$.

Rewriting (1) as

$$(2) \quad f^3 = p(g)e^q + 1 = e^q[p(g) + e^{-q}].$$

Now $p(g) + e^{-q}$ is a function in g with coefficients whose growth is much smaller than that of g . Let's assume that $\beta_1(z), \beta_2(z) \cdots \beta_k(z)$ are k distinct algebraic roots of $p(g) + e^{-q} = 0$. That is

$$(3) \quad p(g) + e^{-q} \equiv c_0(g - \beta_1)^{n_1}(g - \beta_2)^{n_2} \cdots (g - \beta_k)^{n_k},$$

where c_0 is a nonzero constant, and n_1, n_2, \dots, n_k are nonnegative integers. Now if e^{-q} is not a constant, then by taking partial derivative with respect to g on both sides of (3) one can conclude easily that all the n_i ($i=1, 2, \dots, k$) are no greater than one, i.e., $n_i \leq 1$. Furthermore, according to a result of C. Osgood's [7] g can not have two or more algebraic roots which are complete ramified (of multiplicity ≥ 3). Thus we conclude that one and only one of the n_i is equal to 1, the remaining ones being equal to zero. This is impossible.

Hence we conclude that e^{-q} must be a constant. Thus

$$f^3 = c_0(g - \alpha_1)^{n_1}(g - \alpha_2)^{n_2} \cdots (g - \alpha_k)^{n_k},$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants. Since an entire function which takes on two distinct finite values only on a straight line is of order no greater than 1 [1]. It follows from this and the assumption on the zeros of f that the α_i 's must be equal. Hence we have

$$f^3 = c_0(g - \alpha_i)^{n_i}$$

for some constant α_i and integer n_i . By virtue of the assumption that all the zero of f are simple it follows that $n_i = 3$. Hence

$$f = c_0^{1/3}(g - \alpha_i) = ag + b$$

for some constants a and b . This completes the proof of the theorem.

We now prove the existence of an infinite URS.

Theorem 3. *Let $T = \{\xi \mid e^\xi + \xi = 0\}$, then T is URS.*

Proof. Suppose that there are two nonconstant entire functions f and g such that

$$(4) \quad f^{-1}(T) = g^{-1}(T).$$

Then, accordingly, we have

$$(5) \quad e^f + f = (e^g + g)e^\alpha,$$

where α is an entire function.

It follows from this and Nevanlinna's second fundamental theorem for three deficient functions (see e.g., [4, p. 47]), that

$$(6) \quad T(r, e^f) \sim N(r, 1/e^f + f) \sim N(r, 1/e^g + g) \sim T(r, e^g)$$

as $r \rightarrow \infty$; $r \in E, E$ a set of r values of finite linear measure.

By rewriting (5) we have

$$(7) \quad e^f - e^{g+\alpha} - ge^\alpha \equiv -f.$$

This is the so-called Borel type of identity. It is now easy to verify, with (6) in mind, that the conditions for the generalized Borel's lemma of impossibility of certain identity discussed in [5] are satisfied. Con-

sequently, one can derive from (7) that $f - g \equiv c$, a constant. This and (7) yields $f \equiv g$ as claimed.

1.3 Open questions. *Question 1.* Does there exist an infinite NPS, S such that for any nonempty finite set S_1 , $S \setminus S_1$ (in this case $S_1 \subset S$) or $S \cup S_1$ remains an NPS?

Question 2. Given any discrete set S which is an NPS is it always possible to add or remove only a finite set F such that $S \cup F$ or $S \setminus F$ becomes an NPS? Where in the union case some elements will be counted according to its number of appearances in both S and F ?

Question 3. Does there exist a UNPS for an entire function of integral order?

Question 4. To each given number ρ ; $0 \leq \rho \leq \infty$, does there exist a UNPS such that its corresponding entire function has an order equal to ρ ?

An entire function F is called prime if $F \equiv f(g)$ for some entire functions f and g then it implies either f or g must be linear.

Question 5. One can show easily that every zero set of a prime entire function of order less than 1 is not an NPS. How about the zero set of a prime entire function of nonintegral order ≥ 1 ?

Clearly this is not true in general for integral order. For instance, $F(z) = e^{z^2}(e^z - 1)$ is a prime entire function of order 2 and $e^z - 1 \equiv p(g)$ where

$$p(z) = z^n - 1 \quad \text{and} \quad g(z) = e^{z/n}.$$

Acknowledgment. The authors would like to thank the anonymous referee for the suggestion of giving theorem 3 the present much shorter proof.

References

- [1] A. Edrei: Meromorphic functions with three radially distributed values. Trans. Amer. Math. Soc., **78**, 278–293 (1955).
- [2] F. Gross: Factorization of Meromorphic Functions and some Open Problems, Complex Analysis. Lect. Notes in Math., vol. 599, Springer-Verlag, New York (1976).
- [3] G. Gundersen: Meromorphic functions that share three or four values. London Math. Soc., (2) **20**, 457–466 (1979).
- [4] W. K. Hayman: Meromorphic Functions. Oxford University Press (1964).
- [5] G. Hiromi and M. Ozawa: On the existence of analytic mapping between two ultrahyperelliptic surfaces. Kodai Math. Sem. Rep., **17**, 281–306 (1965).
- [6] R. Nevanlinna: Le théorème de Picard-Borel et la théorie des fonctions méromorphes. Paris, Gauthier-Villars (1929).
- [7] C. F. Osgood: Extending Nevanlinna's Ramification Result (to appear).
- [8] L. A. Rubel and C. C. Yang: Interpolation and unavoidable families of meromorphic functions. Michigan Math. J., **20**, 289–296 (1973).