42. The Probabilistic Treatment of Phase Separations in Ising Model with Free Boundary Condition

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Introduction. In this paper we consider the problem of phase separations in Ising model with free boundary condition. As for the pure boundary condition this problem was considered by Minlos and Sinai [1], [2]. Let V be the square in Z^2 with the area $|V|$. Consider the $(+)$-boundary condition and fix the number of minus spins $\rho|V|$ $(0<\rho<1)$. Then they showed that a typical configuration has one connected component of minus spins with the shape of nearly square whose width is about $\rho^{1/2}|V|^{1/2}$.

On the other hand, this fact does not hold in the case of free boundary condition, and the following conjecture was given in [3]; a typical configuration has just one open contour λ which separates V into two parts which are occupied by the opposite phases and λ should be shortest under the condition that V is divided by λ into two regions of volume $\rho|V|$ and $(1-\rho)|V|$. We prove this conjecture positively with respect to the conditional Gibbs measure with free boundary condition.

§1. Ising model with free boundary condition. In this section we give the definition of the conditional Gibbs measure of two-dimensional Ising model with free boundary condition.

Let V be the square in Z^2. Put $\Omega_V=\{+1,-1\}^V$ and $\mathfrak{B}_V=\sigma\{\omega(t)\,;t\in V\}$. For a given configuration $\xi\in\Omega_V$ we draw a unit segment perpendicular to the center of each bond between different spins. Then we have the family of closed lines and open lines. We give these lines the orientations along which we see plus spins on the left side. (See Fig. 1.)

For each line Γ, put $\bar{\Gamma}=(\Gamma,+)$ or $\bar{\Gamma}=(\Gamma,-)$ according that the orientation of Γ is clockwise or anti-clockwise, respectively. We call $\bar{\Gamma}$ open contour if Γ is an open line, and call others closed contours.

It is clear that there is a 1-1 correspondence between the configuration $\xi\in\Omega_V$ and the family of contours $(\bar{\Gamma}_1,\cdots,\bar{\Gamma}_s,\bar{\Delta}_1,\cdots,\bar{\Delta}_k)$, where $\bar{\Gamma}_1,\cdots,\bar{\Gamma}_s$ are closed contours and $\bar{\Delta}_1,\cdots,\bar{\Delta}_k$ are open contours.

The Ising model with free boundary condition is defined by the following probability measure on $(\Omega_V,\mathfrak{B}_V)$,

(1.1) $$P_V(\xi)=Z_V^{-1}\exp\{-\beta(\sum_{i=1}^{s}|\Gamma_i|+\sum_{j=1}^{k}|\Delta_j|)\}$$
$$\xi=(\bar{\Gamma}_1,\cdots,\bar{\Gamma}_s,\bar{\Delta}_1,\cdots,\bar{\Delta}_k)$$
where β^{-1} is proportional to the temperature.

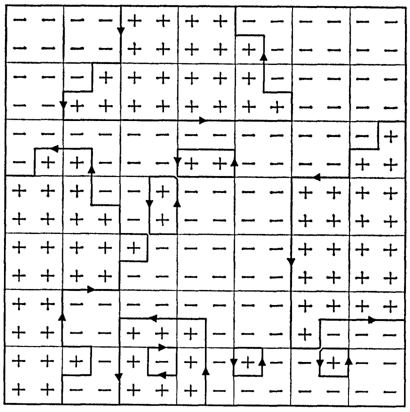

Fig. 1

Next we give the definition of the conditional Gibbs measure. Let $N^-(\xi;V)$ is the number of minus spins in V under the configuration $\xi\in\Omega_V$ and $g(\beta)$ be the function of β satisfying $g(\beta)\downarrow 0$ and $e^{+4\beta}g(\beta)\downarrow 0$ as $\beta\to\infty$. Put $N_\rho^-=\{\xi\in\Omega_V;|N^-(\xi;V)-\rho|V||<g(\beta)|V|\}$, where $0<\rho<1$. Then the conditional Gibbs measure is given by

(1.2) $$P_{V,\rho}(\cdot)=P_V(\cdot\,|N_\rho^-).$$

We prepare some terminologies which will be used in §2. If $|\Gamma|>c_0\ln|V|$, contour $\bar{\Gamma}$ in V is called c_0-large and others are called c_0-small. We call c_0-large open contours and c_0-large closed contours which are not surrounded by any c_0-small closed contour "phase boundary" and denote it by Λ. By the phase boundary V is divided into two parts Θ_+ and Θ_- (+)-phase and (−)-phase respectively. We also denote the maximal connected components of Θ_+ and Θ_- by $\Theta_+^{\max}$ and $\Theta_-^{\max}$ respectively.

§2. **Results.** In this section we state our results rigorously in the following three cases according to the value of ρ.

Case I. $0<\rho<1/4$.

Theorem 1. *If we fix the value of β sufficiently large, then the following two properties are obtained;*

(1) $\lim_{V\uparrow Z^2} P_{V,\rho}(|\Lambda|>(2\rho^{1/2}+k/\beta)|V|^{1/2})=0$ *if* $k>4/c_0$

(2) $\lim_{V\uparrow Z^2} P_{V,\rho}(||\Theta_-^{\max}|-\rho|V||>k(\beta)|V|)=0$

where $k(\beta)=(5k_0/3\beta)^2$ *and* $k_0=4/c_0+1$.

From this theorem we can see that a configuration in V randomly chosen from N_ρ^- satisfies the following properties i) $|\Lambda|\sim 2\rho^{1/2}|V|^{1/2}$ ii) $|\Theta_-^{\max}|\sim\rho|V|$ asymptotically with probability one as $V\uparrow Z^2$. This

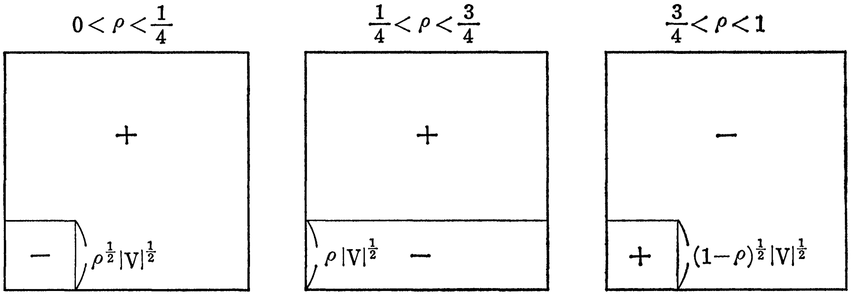

Fig. 2

property means that the shape of $\Theta_-^{\max}$ is nearly square with the side length about $\rho^{1/2}V^{1/2}$ and $\Theta_-^{\max}$ lies in one of the corners. (See Fig. 2.)

Case II. $1/4<\rho<3/4$.

Theorem 2. *For fixed sufficiently large β, we have the following properties*

(1) $\lim_{V\uparrow Z^2} P_{V,\rho}(|\Lambda|>(1+k/\beta)|V|^{1/2})=0$ *if* $k>4/c_0$

(2) $\lim_{V\uparrow Z^2} P_{V,\rho}(||\Theta_-^{\max}|-\rho|V||>k(\beta)|V|)=0$

where $k(\beta)$ is the same function given in Theorem 1.

This theorem means that there exists one open contour λ starting from one side to the opposite side whose length is nearly $|V|^{1/2}$, and that V is divided by λ into two regions of volume $\rho|V|$ and $(1-\rho)|V|$. (See Fig. 2.)

Case III. $3/4<\rho<1$.

In this case we have the similar results to Theorem 1.

Theorem 3. *For fixed sufficiently large β, we have*

(1) $\lim_{V\uparrow Z^2} P_{V,\rho}(|\Lambda|>(2(1-\rho)^{1/2}+k/\beta)|V|^{1/2})=0$ *if* $k>4/c_0$

(2) $\lim_{V\uparrow Z^2} P_{V,\rho}(||\Theta_-^{\max}|-\rho|V||>k(\beta)|V|)=0$

where $k(\beta)$ is the same function given in Theorem 1.

We give the sketch of proof of Theorem 1. The proof of Theorems 1 and 2 is very similar to Theorem 1. The key point of the proof is the following estimate of $P_V(N_\rho^-)$ from below. We treat only the case of $0<\rho<1/4$.

Lemma 1. *If we fix β sufficiently large, then we have the following estimate for sufficiently large V,*

$$P_V(N_\rho^-)>c\exp[-(2\rho^{1/2}\beta+m(\beta))|V|^{1/2}]$$

where $m(\beta)\sim\exp(-4\beta)$.

As for the length of phase boundaries, we have the following estimate by the same way as in the case of pure boundary condition.

Lemma 2. $P_V(|\Lambda|>T)<(4/\ln|V|)\exp[-(\beta-2/c_0)T]$.

From Lemmas 1 and 2 we have the first assertion of Theorem 1.

By using the following lemma proved in the paper of Minlos and Sinai [1], we can prove the second assertion of Theorem 1.

Lemma 3 (Minlos and Sinai). *If* $|\Theta_{\pm}|>k|V|$, *then*

$$P_{\Theta_{\pm},\pm,c_0}(|N_{\pm}(\xi\,;\Theta_{\pm})-\rho^{**}|\Theta_{\pm}|>t|\Theta_{\pm}|^{3/4})<c\exp[-q(\beta)t^2k^{1/2}|V|^{1/2}]$$

where $q(\beta)\sim\exp(-4\beta)$ *and* $P_{\Theta_{\pm},\pm,c_0}(\ )$ *is the conditional measure under the condition that all outer contours in* $\Theta_{\pm}$ *are* c_0*-small.*

References

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