31. Classification of Projective Varieties of Δ-Genus One

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Introduction. Let V be a subvariety (=irreducible reduced closed subscheme) of a projective space $\boldsymbol{P}^N$ defined over an algebraically closed field $\mathfrak{K}$ of any characteristic. Set $n=\dim V$, $d=\deg V$ and $m=\operatorname{codim} V=N-n$. In this note we always assume that the restriction mapping $H^0(\boldsymbol{P}^N, \mathcal{O}(1))\to H^0(V, L)$ is bijective, where $L=\mathcal{O}_V(1)$. Then $\Delta=d-m-1=n+d-h^0(V,L)$ is the Δ-genus of the polarized variety (V,L) (cf. [1] etc.).

It is well-known that $\Delta\geqq 0$ for every V as above. Moreover, we have the following

Theorem 0 (see, e.g., [1] if char$(\mathfrak{K})=0$ and [4] in general). *If $\Delta=0$, then V is one of the following types*:

1) $(\boldsymbol{P}^n, \mathcal{O}(1))$.

2) *A hyperquadric.*

3) *A rational scroll. This means that $(V,L)\cong(\boldsymbol{P}(E), \mathcal{O}(1))$ for an ample vector bundle E on $\boldsymbol{P}^1$.*

4) *A Veronese surface $(\boldsymbol{P}^2, \mathcal{O}(2))$ in $\boldsymbol{P}^5$.*

5) *A generalized cone* (this means that the set of the vertices may be a linear space of positive dimension) *over a projective manifold of one of the above types* 2)–4).

In this note we consider the case $\Delta=1$. Details and proofs will be published elsewhere.

As for non-singular varieties, we have the following

Theorem I (cf. [2] [3] and [4]). *Let V be a projective non-singular variety as above with $\Delta=1$. Then the dualizing sheaf ω_V is isomorphic to $\mathcal{O}_V(1-n)$. Moreover, if $n\geqq 3$, then V is one of the following types*:

1) *A hypercubic.* $d=3$.

2) *A complete intersection of two hyperquadrics.* $d=4$.

3) *A linear section of the Grassmann variety parametrizing lines in $\boldsymbol{P}^4$, embedded by the Plücker coordinate.* $d=5$ *and* $n\leqq 6$.

4) (A hyperplane section of) *the Segre variety $\boldsymbol{P}^2\times\boldsymbol{P}^2$ in $\boldsymbol{P}^8$.* $d=6$.

5) *The Segre variety $\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ in $\boldsymbol{P}^7$.* $d=6$.

6) *The blowing-up of $\boldsymbol{P}^3$ at a point.* $d=7$.

7) *Veronese threefold $(\boldsymbol{P}^3, \mathcal{O}(2))$ in $\boldsymbol{P}^9$.* $d=8$.

Remark. When $n=2$, V is what is called a del Pezzo surface. V is obtained from $\boldsymbol{P}^2$ by blowing-up at $(9-d)$ points on it, unless $V\cong \boldsymbol{P}^1\times\boldsymbol{P}^1$. In particular, $d\leqq 9$.

Now we consider singular varieties. First we present a couple of trivial examples.

Let W be a subvariety of a hyperplane H in $\boldsymbol{P}^N$ such that the mapping $H^0(H,\mathcal{O}_H(1))\to H^0(W,\mathcal{O}_W(1))$ is bijective and that $\Delta(W,\mathcal{O}_W(1))=1$. Take a point v off H and let V be the union of all the lines passing v and intersecting W. Then V is a variety with $\Delta=1$ such that $H^0(\boldsymbol{P}^N,\mathcal{O}(1))\to H^0(V,\mathcal{O}_V(1))$ is bijective. In this case we say that V is a cone over W.

Any hypercubic has the property $\Delta=1$. The same is true for any complete intersection of two hyperquadrics.

From now on, we assume that V is none of the above types—not a cone, not a hypercubic, not a complete intersection of two hyperquadrics.

For the convenience of the statements about possible singularities of V, we make several definitions and introduce notations.

Definition. Let x be an isolated singular point of a variety X. We consider the type of this singularity according to the completion of the local ring $\mathcal{O}_{X,x}$.

1) x is said to be of type (N^s) if there are two analytic branches of X at x, both of which are non-singular and of dimension s, and if they intersect transversally at x.

2) x is said to be of type (C^s) if the normalization X' of X is non-singular and of dimension s, the mapping $f: X'\to X$ is set-theoretically bijective and if $\mathrm{Coker}\,(\mathcal{O}_X\to f_*\mathcal{O}_{X'})\cong\mathcal{O}_x\cong\mathfrak{K}$.

3) x is said to be of type (A_k) if $\dim X=2$ and if x is the hypersurface singularity defined by the equation $uv=w^k$.

4) x is said to be of type (Q^s) if the singularity is the same as that of the vertex of the affine cone of a non-singular hyperquadric of dimension $s-1$.

Remark. $(\mathrm{Q}^1)=(\mathrm{N}^1)$ and $(\mathrm{Q}^2)=(\mathrm{A}_1)$ as types of singularities. (N^1) is a node of a curve. (C^1) is a simple cusp.

Definition. Let S be the singular locus of a variety Y and let x be a simple point of S. Let r be the dimension of S at x. Taking r general hyperplane sections passing x successively we obtain a linear section X of Y which has an isolated singularity at x. If this is one of the above types $(*)$, we say that Y has a singularity of type $(*)$ at x, or that x is a singular point of Y of type $(*)$.

Definition. Let T be a connected component of the singular locus of a variety Y. We say that T is of type

$\boldsymbol{P}^r(*)$, if T is an r-dimensional linear subspace of $\boldsymbol{P}^N$ and if Y has a singularity of type $(*)$ at every point on T;

$(\boldsymbol{P}^r, H)(*, **)$, if T is a linear subspace of dimension r and if there exists a hyperplane H on T such that the singularity of Y is of type $(**)$ at every point on H and is of type $(*)$ at evrey point on $T-H$;

$(\boldsymbol{P}^r, 2H)(*, **)$, if T is a linear subspace of dimension r and if there exists a non-singular hyperquadric Q on T such that the singularity of Y is of type $(**)$ at every point on Q and is of type $(*)$ at every point on $T-Q$.

Type $\boldsymbol{P}^0(*)$ is denoted simply by $(*)$.

Definition. We say that V has a singularity of type $(*_1)\coprod\cdots\coprod(*_q)$ if the singular locus consists of q connected components $S_1, \cdots, S_q$ such that S_j is of type $(*_j)$ for each $j=1, \cdots, q$.

Theorem II. *Let V be a projective variety with $\Delta=1$ as before* (hence, not a cone, not a hypercubic, not a complete intersection of two hyperquadrics). *Suppose that V is not normal and let $f: V'\to V$ be the normalization of V. Then*

a) *V' is non-singular and $\Delta(V', f^*L)=0$. Moreover, V' is of the type* 3) *in Theorem* 0.

b) *V has a singularity of one of the following types:*

(N^n), (C^n), $(\boldsymbol{P}^1, H)(\mathrm{N}^{n-1}, \mathrm{C}^{n-1})$; *these three are possible in any characteristic,*

$(\boldsymbol{P}^1, 2H)(\mathrm{N}^{n-1}, \mathrm{C}^{n-1})$, $(\boldsymbol{P}^2, 2H)(\mathrm{N}^{n-2}, \mathrm{C}^{n-2})$; *these are possible only when* char $(\mathfrak{K})\neq 2$,

$\boldsymbol{P}^1(\mathrm{C}^{n-1})$, $(\boldsymbol{P}^2, H)(\mathrm{N}^{n-2}, \mathrm{C}^{n-2})$; *these are possible only when* char $(\mathfrak{K})=2$.

In particular, the singular locus of V is connected and is a linear space of dimension $\leqq 2$.

Theorem III. *Let V be a singular projective variety with $\Delta=1$ as before. Suppose that V is normal. Then*

a) *V is locally Gorenstein and $\omega_V=\mathcal{O}_V(1-n)$.*

b) $(n, d)=(\dim V, \deg V)$ *can take only the following values:* $(2,8)$, $(2,7)$, $(2,6)$, $(2,5)$, $(3,6)$, $(3,5)$, $(4,6)$, $(4,5)$ *and* $(5,5)$.

c) *The possible singularities of V with given (n, d) is one of the following types.*

Case $(2,8)$: (A_1).

Case $(2,7)$: (A_1).

Case $(2,6)$: (A_1), $(\mathrm{A}_1)\coprod(\mathrm{A}_1)$, (A_2), $(\mathrm{A}_1)\coprod(\mathrm{A}_2)$.

Case $(2,5)$: (A_1), $(\mathrm{A}_1)\coprod(\mathrm{A}_1)$, (A_2), $(\mathrm{A}_1)\coprod(\mathrm{A}_2)$, (A_3), (A_4).

Case $(3,6)$: (Q^3), $\boldsymbol{P}^1(\mathrm{A}_1)$, $(\mathrm{Q}^3)\coprod\boldsymbol{P}^1(\mathrm{A}_1)$, $\boldsymbol{P}^1(\mathrm{A}_2)$.

Case $(3,5)$: (Q^3), $(\mathrm{Q}^3)\coprod(\mathrm{Q}^3)$, $(\mathrm{Q}^3)\coprod(\mathrm{Q}^3)\coprod(Q^3)$, $(\mathrm{Q}^3)\coprod(\boldsymbol{P}^1, H)(\mathrm{A}_1, \mathrm{A}_2)$, $(\boldsymbol{P}^1, H)(\mathrm{A}_1, \mathrm{A}_2)$, $(\boldsymbol{P}^1, H)(\mathrm{A}_2, \mathrm{A}_3)$, $(\boldsymbol{P}^1, H)(\mathrm{A}_2, \mathrm{A}_4)$.

Case (4, 6): $\boldsymbol{P}^2(\mathrm{A}_1)$.

Case (4, 5): $\boldsymbol{P}^1(\mathrm{Q}^3)$, $\boldsymbol{P}^1(\mathrm{Q}^3)\coprod \boldsymbol{P}^1(\mathrm{Q}^3)$, $(\boldsymbol{P}^2, H)(\mathrm{A}_1, \mathrm{A}_2)$.

Case (5, 5): $\boldsymbol{P}^2(\mathrm{Q}^3)$.

In particular, V has only rational hypersurface singularities, and every connected component of its singular locus is a linear space of dimension $\leqq 2$. There is no special phenomenon in case char$(\mathfrak{K})=2$.

Outline of proofs of Theorems II and III. Take a singular point v of V. Let W be the closure of the union of all the lines connecting v and another point on V. Then $\dim W=n+1$ and $\deg W \leqq d-2$. Hence $\Delta(W, \mathcal{O}_W(1))=0$ and $\deg W=d-2$. By virtue of Theorem 0, W is a generalized cone over a manifold M of one of the types 2)–4) in Theorem 0. Let R be the set of vertices of W. Let $\tilde{W}$ be the blow-up of W with center R and let $\tilde{V}$ be the strict transform V on $\tilde{W}$. Then $\tilde{W}$ is a $\boldsymbol{P}^{r+1}$-bundle over M associated with the locally free sheaf $\mathcal{O}_M(1) \oplus\mathcal{O}_M\oplus\cdots\oplus\mathcal{O}_M$, where $r=\dim R$. $\tilde{V}$ is a divisor on $\tilde{W}$. We analyze all the possible cases according to the class of $\tilde{V}$ in $\mathrm{Pic}(\tilde{W})$.

References

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