29. *Functional Equations and Hypoellipticity*

By Akira TSUTSUMI*) and Shigeru HARUKI**)

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1982)

1. In this note we investigate the problem whether all the continuous or all the locally integrable solutions of certain functional equations are C^∞ or not. An affirmative answer to this problem under weak regularity assumptions on the equations enables one to make easier to find all the continuous or sometimes all the locally integrable solutions of the equations. Because we can use a powerful means-differentiation (see [1], [2]). The aim of this note is to give a general method for this problem. We consider the functional equation of the unknown $f(x)$:

(1.1) $\sum_{j=1}^{k} a_j(x,t)f(h_j(x,t))=F(x, f(l_1(x)), \cdots, f(l_s(x)))+b(x,t)$

where $x \in R^n$ and $t \in R^r$, with the assumptions followed.

(A.1) $a_j(x,t)$, $b(x,t) \in C^\infty(R^n)$ for every fixed t from an open set $\omega \subseteq R^r$, $j=1, \cdots, k$,

(A.2) $a_j(x,t)$, $b(x,t) \in C^m(R^n \times \omega)$, $j=1, \cdots, k$,

(A.3) the mappings $x \mapsto y=h_j(x,t)$ are diffeomorphisms in R^n for every fixed $t \in \omega$, $j=1, \cdots, k$,

(A.4) $h_j(x,t) \in C^m(R^n \times \omega)$ and its inverse $h_j^{-1} \in C^m$, $j=1, \cdots, k$,

(A.5) $F(x, z_1, \cdots, z_s) \in C(R^{n+s})$,

(A.6) $l_j(x) \in C(R^n)$, $j=1, \cdots, s$.

A locally integrable function $f(x)$, $x \in R^n$, is said to be *a solution of* (1.1) *in the sense of distribution* (or *a distribution solution*) if

(1.2)
$$\sum_{j=1}^{k}\int_{R^n} a_j(x,t)f(h_j(x,t))\phi(x)dx = \int_{R^n} F(x, f(l_1(x)), \cdots, f(l_s(x)))\phi(x)dx + \int_{R^n} b(x,t)\phi(x)dx$$

for each $\phi \in \mathcal{D}$ and every fixed $t \in \omega$. We can write it briefly

(1.3)
$$\sum_{j=1}^{k}(a_j(x,t)f(h_j(x,t)), \phi(x))_x = (F(x, f(l_1(x)), \cdots, f(l_s(x))), \phi(x))_x + (b(x,t), \phi(x))_x.$$

Let be

$$\partial_x=\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \quad \partial_t=\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_r}\right), \quad \alpha=(\alpha_1, \cdots, \alpha_n)$$

$$\partial_x^\alpha=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \qquad \partial_t^\beta=\left(\frac{\partial}{\partial t_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{\partial t_r}\right)^{\beta_n},$$

$$D_x^\alpha=(-i)^{|\alpha|}\partial_x^\alpha,$$

---

*) Okayama University.

**) Okayama University of Science.

$|\alpha|=\alpha_1+\cdots+\alpha_n$, $|x|=(\sum_{j=1}^k x_j^2)^{1/2}$.
For a linear partial differential operator of order m

$$P(x, D_x)=\sum_{|\alpha|\le m} a_\alpha(x)D_x^\alpha$$

we set

$$P(x, \xi)=\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha, \quad P_{(\beta)}^{(\alpha)}(x, \xi)=\partial_\xi^\alpha D_x^\beta P(x, \xi), \quad \xi\in R^n.$$

$P=P(x, D_x)$ defined in an open set $\Omega\subseteq R^n$ with C^∞ coefficients is said to be *hypoelliptic in* Ω if for $u\in\mathcal{D}'(\Omega)$ and any open subset Ω' of $Pu\in C^\infty(\Omega')$ leads to $u\in C^\infty(\Omega')$. The notion of hypoellipticity comes from the problem whether a distribution solution of the partial differential equation $Pu=f$ is a classical solution or not (see [7]). Since then many sufficient conditions for hypoellipticity have been obtained (see for example [4]–[6], [9]). We cite here one of the most general criterion for hypoellipticity which will be applied in § 3.

2. Composing the results in [4] and [6] the following criterion for hypoellipticity is obtained. Let be

$m=(m_1, \cdots, m_n)$, $\overline{m}=\max(m_j)$, $|\alpha : m|=\sum_{j=1}^n \alpha_j/m_j$,
$x=(x', x'')$, $\xi=(\xi', \xi'')\in R^\nu\times R^{n-\nu}$, $x=(x', \tilde{x}'', \tilde{\tilde{x}}'')\in R^\nu\times R^{\nu'}\times R^{n-(\nu+\nu')}$,
$(1\le\nu\le\nu'\le n)$, $\gamma=(\gamma_1, \cdots, \gamma_{\nu'}, 0, \cdots, 0)$, $(x', \tilde{x}'')^\gamma=x_1^{\gamma_1}\cdots x_{\nu'}^{\gamma_{\nu'}}$.

Consider the partial differential operator of the form

$$P(x, D_x)=\sum a_{\alpha\gamma}(x)(x', \tilde{x}'')^\gamma D_x^\alpha \tag{2.1}$$

where $a_{\alpha\gamma}(x)\in C^\infty(\Omega)$ and the summation $\sum$ is done for $\{\alpha, \gamma\}$ such that $|\alpha : m|=1$ and $(\rho, \alpha)\le(\sigma, \gamma)\le(\rho, \alpha)-\overline{m}$ are satisfied for some fixed indices ρ and σ. And we define from (2.1)

$$\mathcal{P}(x', \tilde{x}'', D_x)=\sum\nolimits_0 a_{\alpha\gamma}(0)(x', \tilde{x}'')^\gamma D_x^\alpha \tag{2.2}$$

where the summation $\sum_0$ is done for $\{\alpha, \gamma\}$ such that $(\rho, \alpha)=(\sigma, \gamma)-\overline{m}$ is satisfied.

Consider the following conditions:

(C.1) There exist two multi-indices $\rho=(\rho_1, \cdots, \rho_n)$ and $\sigma=(\sigma_1, \cdots, \sigma_n)$ such that

(i) $\rho_j=\sigma_j=\overline{m}/m_j$ for $j\ge\nu$,

(ii) $\rho_j>\sigma_j\ge 0$, $m_j\rho_j\ge\overline{m}$ for $\nu<j\le n$,

(iii) $\sigma_j=0$ for $j\ge\nu'+1$,

(C.2) $\mathcal{P}(\lambda^{-\sigma}(x', \tilde{x}''), \lambda^\rho\xi)=\lambda^{\overline{m}}\mathcal{P}(x', \tilde{x}'', \xi)$, $(\lambda>0)$

where $\lambda^{-\sigma}(x', \tilde{x}'')=(\lambda^{-\sigma_1}x_1, \cdots, \lambda^{-\sigma_{\nu'}}x_{\nu'})$ and $\lambda^\rho\xi=(\lambda^{\rho_1}\xi_1, \cdots, \lambda^{\rho_n}\xi_n)$.

We define from (2.2)

$$\mathcal{P}_0(x', \tilde{x}'', D_x)=\sum_{|\alpha:m|=1} a_{\alpha\gamma}(0)(x', \tilde{x}'')^\gamma D_x^\alpha. \tag{2.3}$$

(C.3) $\mathcal{P}_0(x', \tilde{x}'', \xi)=0$ for $(x', \tilde{x}'')\ne 0$ and $\xi\ne 0$: that is $\mathcal{P}_0(x', \tilde{x}'', D_x)$ is semi-elliptic at $(x, \tilde{x}'')\ne 0$.

(C.4) For any $\tilde{x}''$ and $\xi''=(\xi_{\nu+1}, \cdots, \xi_n)$ with $|\xi''|=1$ the equation $\mathcal{P}(x', \tilde{x}'', D_{x'}, \xi'')v=0$ does not have any non-trivial solution in $\mathcal{S}(R_{x'}^\nu)$.

A sufficient condition for hypoellipticity which we use is the following. *If the operator* (2.1) *satisfies* (C.1)–(C.4) *and* $\max_{\nu\le j\le n}\{\sigma_j\}=\min_{\nu<j,l\le n}\{m_j\rho_j/m_l\}$ *is satisfied, then the operator* (2.1) *is hypoelliptic.*

The following examples illustrate this criterion.

Example. (i) Let be the operator $P=(-\Delta_{x'})^l+|x'|^{2k}(-\Delta_{x''})^m$, in Ω containing the origin. When we take $\rho_1=\cdots=\rho_\nu=\sigma_1=\cdots=\sigma_\nu=\overline{m}/l$, for $\overline{m}=\max\{m, l\}$ and $\rho_{\nu+1}=\cdots=\rho_n=(k/l+1)\overline{m}/m$, $\sigma_{\nu+1}=\cdots\sigma_n=0$, this operator satisfies the criterion. Thus P is hypoelliptic in Ω.

(ii) Consider the operater $P_\pm=D_{x_1}\pm ix_1^kD_{x_2}^l$ in R^2. Let be $\rho_1=\sigma_1=1$, $\rho_2=k+1$, $\sigma_2=0$. If "k is even" or "k is odd and l is even", then P_+ is hypoelliptic. If k is even, then P_- is hypoelliptic.

3. **Theorem.** *Suppose that the equation* (1.1) *satisfies* (A.1)–(A.6). *And suppose there exist a $t^0\in\omega$ such that $h_i(x, t^0)=x$ for $i=1, \cdots, k$, and a multi-index q ($|q|\le m$) such that the partial differential equation*

$$\partial_t^q\left(\textstyle\sum_{j=1}^k a_j(x, t)f(h_j(x, t))\right)\big|_{t=t^0}=0 \tag{3.1}$$

where ∂_t^q operates formally, is hypoelliptic in R^n.

Then every continuous solution of (1.1) *is $C^\infty(R^n)$, and every locally integrable solution is equal to a function of $C^\infty(R^n)$ almost everywhere.* It is known that there are examples in which the resulting equation (3.1) is elliptic or hypoelliptic with constant coefficients (see [3], [8]). Therefore we show examples in which (3.1) are degenerate hypoelliptic as new results.

Example. (i) The equation

$$\begin{aligned} &f(x_1-t, x_2)+f(x_1+t, x_2)+x_1^2f(x_1, x_2-t)+x_1^2f(x_1, x_2+t)\\ &\quad=2f(x_1, x_2)+2x_1^2f(x_1, x_2)+2[f(x_1, x_2)]^2+[f(x_1+x_2, x_1-x_2)]^2\\ &\qquad-[f(x_1, x_1)]^2-[f(x_2, x_2)]^2 \end{aligned}$$

satisfies the assumptions of Theorem. Differentiating twice in t and setting $t=0$ we obtain

$$\partial_{x_1}^2f(x_1, x_2)+x_1^2\partial_{x_2}^2f(x_1, x_2)=0.$$

By the example (i) of § 2 this is hypoelliptic.

(ii) Differentiating the equation

$$f(x_1+t, x_2)+ix_1^4f(x_1, x_2+t^2)-2f(x_1, x_2)=0$$

in t and setting $t=0$ we obtain

$$\partial_{x_1}f(x_1, x_2)+ix_1^4\partial_{x_2}^4f(x_1, x_2)=0,$$

which is hypoelliptic by the example (ii) of § 2.

As the both resulting partial differential equations in the above are elliptic outside of the origin, we remark that the criterion for hypoellipticity works essentially in the neighborhood of the origin.

References

[1] J. Aczél: Lectures on Functional Equations and their Applications. Academic Press, New York and London (1966).

[2] J. Aczél, S. Haruki, M. Mckiernan, and G. Sakovic: General and regular solutions of functional equations characterizing harmonic polynomials. Aequations Math., 1, 37–53 (1968).

[3] I. Fenyö: Über eine Functionalgleichung. Math. Nachr., **31**, 103–109 (1966).
[4] V. V. Grushin: Hypoelliptic differential equations and pseudo-differential equations with operator-valued symbols. Math. USSR Sb., **17**, 497–514 (1972).
[5] L. Hörmander: Pseudodifferential operators and hypoelliptic equations. Proc. Symp. on Singular Integrals, A. M. S., no. 10, pp. 138–183 (1967).
[6] H. Kumano-go and K. Taniguchi: Oscillatory integrals of symbols of pseudodifferential operators on R^n and operators of Fredholm type. Proc. Japan Acad., **49**, 397–402 (1973).
[7] L. Schwartz: Théorie des distributions, I & II. Hermann, Paris (1966).
[8] H. Świątak: The regularity of the locally integrable and continuous solutions of nonlinear functional equations. Trans. Amer. Math. Soc., **221**, 97–118 (1976).
[9] A. Tsutsumi: On the asymptotic behavior of resolvent kernels and spectral functions for some class of hypoelliptic operators. J. Diff. Equ., **18**, 366–385 (1975).