80. The τ Function of the Kadomtsev-Petviashvili Equation

Transformation Groups for Soliton Equations. I

By Masaki KASHIWARA and Tetsuji MIWA
Research Institute for Mathematical Sciences, Kyoto University

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The notion of τ function was first introduced by Sato, Miwa and Jimbo in a series of papers on holonomic quantum fields [1]. There the τ functions were simply expressed as the expectation values of field operators which belong to the Clifford group of free fermions. Then by several authors [2]-[4] the concept was generalized and exploited in the study of monodromy and spectrum preserving deformations. They also showed that τ functions are nothing other than the dependent variables used by Hirota in this theory of bilinear equations [5].

This is the first in a series of papers [6], [7] by the present authors, E. Date and M. Jimbo, which aims at a further study of τ functions and soliton equations.

The main results in the present paper are the following. a) We construct a Clifford operator $\varphi(x)$ so that for any even Clifford group element g the expectation value $\tau(x)=\langle\varphi(x)g\rangle$ gives us a solution to the hierarchy of the KP (Kadomtsev-Petviashvili) equations in Hirota's bilinear form. b) Define polynomials $p_j(x)$ $(j=0,1,2,\cdots)$ by

$$\exp\left(\sum_{j=1}^{\infty} k^j x_j\right)=\sum_{j=0}^{\infty} p_j(x)k^j.$$

The KP hierarchy contains the following infinite number of bilinear equations:

$$\det\begin{bmatrix} p_{f_1+1}\left(-\frac{\tilde{D}}{2}\right) & p_{f_1+1}\left(\frac{\tilde{D}}{2}\right) & \cdots p_{f_1+m-1}\left(\frac{\tilde{D}}{2}\right) \\ p_{f_2}\left(-\frac{\tilde{D}}{2}\right) & p_{f_2}\left(\frac{\tilde{D}}{2}\right) & \cdots p_{f_2+m-2}\left(\frac{\tilde{D}}{2}\right) \\ \vdots & \vdots & \vdots \\ p_{f_m-m+2}\left(-\frac{\tilde{D}}{2}\right) & p_{f_m-m+2}\left(\frac{\tilde{D}}{2}\right) & \cdots p_{f_m}\left(\frac{\tilde{D}}{2}\right) \end{bmatrix} \tau(x)\cdot\tau(x)=0,$$

$(f_1\geq f_2\geq\cdots\geq f_m\geq 1,\ m\geq 2,\ \tilde{D}=(D_1,\ D_2/2,\ D_3/3,\ \cdots))$.

Our work is deeply related to the recent progress [8] by M. Sato and Y. Sato on the structure of τ functions for the KP hierarchy. In fact, our starting point was Sato's lecture [9] in which he claimed that the space of solutions to the KP hierarchy is neatly parametrized by the Grassmann manifold of infinite dimensions.

In § 1 we give the definition of the τ function for the KP hierarchy. This part is due to an unpublished work of M. Jimbo, M. Sato and one of the present authors (T. M.). In § 2 we give an expression for the τ functions of the KP hierarchy by means of the Clifford operators used in [10]. In § 3 we give the explicit forms of Hirota's bilinear equations for them using a transformation introduced in [8]. In § 4, as a corollary of the operator expression given in § 2, we show that the characters of the general linear group are τ functions for the KP hierarchy; that is one of the astonishing results of M. and Y. Sato [9].

We thank M. Sato for his explanation on his recent work with Y. Sato before publication. We benefited from discussions with M. Jimbo.

§ 1. We denote by $x=(x_1, x_2, \cdots)$ an infinite number of independent variables. The hierarchy of higher-order KP equations is a system of non-linear partial differential equations with unknown functions $b_{lm}(x)$ $(l=2,3,\cdots; 0\leq m\leq l-2)$ obtained as the compatibility condition for the following system of linear partial differential equations for a wave function $w(x)$.

$$(1) \qquad \frac{\partial w(x)}{\partial x_l}=\left(\frac{\partial^l}{\partial x_1^l}+\sum_{m=0}^{l-2} b_{lm}(x)\frac{\partial^m}{\partial x_1^m}\right)w(x) \qquad l=2,3,\cdots.$$

We set

$$(2) \qquad \xi(x,k)=\sum_{l=1}^{\infty} k^l x_l.$$

Assume the system (1) is compatible. Then, by a suitable change of dependent and independent variables, we can find a formal solution $w(x,k)$ to (1), containing a spectral parameter k, of the form

$$(3) \qquad w(x,k)=\hat{w}(x,k)\exp\xi(x,k),$$

$$(4) \qquad \log\hat{w}(x,k)=\sum_{l=1}^{\infty} t_l(x)k^{-l}.$$

A formal solution is uniquely determined by (1) up to a factor depending on k but independent of x. If we fix a formal solution, the τ function is consistently defined up to a constant factor by the following:

$$(5) \qquad \frac{\partial}{\partial x_l}(\log\tau)=-lt_l(x)-\sum_{m=1}^{l-1}\frac{\partial t_{l-m}(x)}{\partial x_m}.$$

We set $\tilde{\partial}=(\partial/\partial x_1, (1/2)(\partial/\partial x_2), (1/3)(\partial/\partial x_3), \cdots)$ and define a modified τ function $\tau_{[k]}(x)$ by

$$(6) \qquad \tau_{[k]}(x)=e^{\xi(x,k)}e^{-\xi(\tilde{\partial},k^{-1})}\tau(x)=\tau\left(x_1-\frac{1}{k}, x_2-\frac{1}{2k^2}, \cdots\right)e^{\xi(x,k)}.$$

Then the system (5) is rewritten as a formal identity which directly connects $w(x,k)$ with $\tau(x)$;

$$(7) \qquad w(x, k)=\tau_{[k]}(x)/\tau(x).$$

Note that a change of $\tau(x)$ by the exponential of a linear function gives a multiplication to $w(x, k)$ by a function in k.

If $l=1$ in (5), we have

$$(8) \qquad b_{20}(x)=2(\partial^2/\partial x_1^2)\log \tau(x).$$

Thus the introduction of $\tau(x)$ by (5) gives a generalization of Hirota's dependent variable transformation [5] for the total hierarchy. Namely, if the system (1) is compatible, $\tau(x)$ satisfies infinitely many Hirota bilinear differential equations. Let us count the degree of D_l as l. Note that if $P(-D)=-P(D)$ we have $P(D)f(x)\cdot f(x)=0$ for any function $f(x)$. The equation of the lowest degree among non trivial ones is the following [11].

$$(9) \qquad (D_1^4+3D_2^2-4D_1D_3)\tau(x)\cdot\tau(x)=0.$$

In [8], M. Sato and Y. Sato listed the bilinear equations satisfied by $\tau(x)$ up to degree 11, and conjectured that the number of bilinear equations of degree n is equal to $p(n-1)$, where $p(n)$ is the partition function; $p(n)=\#\{(n_1, \cdots, n_s)|n_\sigma$: integer s.t. $1\leq n_1\leq n_2\leq\cdots\leq n_s$ and $\sum_{\sigma=1}^s n_\sigma=n\}$.

In § 3 we propose a refined version of their conjecture by giving explicit forms of bilinear equations for every degree.

§ 2. Now we prepare some notations on characters of the general linear group. Following [9] we set $x_l=\mathrm{trace}\, g^l/l$, where g is an element of $GL(N, \boldsymbol{C})$ for a sufficiently large N. For any Young diagram $Y=(f_1, \cdots, f_m)$ we denote by $\chi_Y(x)$ its character evaluated at g. This is a polynomial in x which is independent of the choice of N larger than $f_1+\cdots+f_m$. We denote by $p_l(x)$ the l-th symmetric tensor (i.e. $Y=(l)$) and by $q_l(x)$ the l-th skew-symmetric tensor (i.e. $Y=(1^l)=(\overbrace{1\cdots 1}^{l})$). For $l<0$ we understand $p_l(x)$ and $q_l(x)$ to be zero then we have

$$(10) \qquad e^{\xi(x,k)}=\sum k^l p_l(x),$$

and $q_l(-x)=(-)^l p_l(x)$.

Now we denote by ψ_m and ψ_m^* $(m\in \boldsymbol{Z})$ free fermions such that $[\psi_m, \psi_n^*]_+=\delta_{mn}$, $[\psi_m, \psi_n]_+=0$ and $[\psi_m^*, \psi_n^*]_+=0$. We also use their Fourier transforms ($\underline{dk}=dk/2\pi ik$):

$$(11) \qquad \begin{aligned}&\psi(k)=\sum_{n\in\boldsymbol{Z}}\psi_n k^n,\ \psi^*(k)=\sum_{n\in\boldsymbol{Z}}\psi_n^* k^{-n},\\ &\psi_n=\int \underline{dk}\, k^{-n}\psi(k),\ \psi_n^*=\int \underline{dk}\, k^n\psi^*(k).\end{aligned}$$

We define the Hamiltonian for the x-flows by

$$(12) \qquad H(x)=\sum_{l\geq 1} x_l\psi_n\psi_{n+l}^*=\int \underline{dk}\,\xi(x, k)\psi(k)\psi^*(k)$$

and set

$$\begin{aligned}\psi_n(x)&=e^{H(x)}\psi_n e^{-H(x)}=\sum_{\nu\in \boldsymbol{Z}}\psi_{n-\nu}p_\nu(x),\\ \psi_n^*(x)&=e^{H(x)}\psi_n^* e^{-H(x)}=\sum_{\nu\in \boldsymbol{Z}}\psi_{n+\nu}^* p_\nu(-x).\end{aligned}\tag{13}$$

We introduce a Clifford operator $\varphi(x)=:e^{\rho(x)}:$ by

$$\begin{aligned}\rho(x)&=\sum_{l\leq -1}\psi_l(-x)\psi_l^*(-x),\\ &=\iint \underline{dk_1}\,\underline{dk_2}\,\frac{1}{1-k_1/k_2}\,\frac{e^{\xi(x,k_2)}}{e^{\xi(x,k_1)}}\,\psi(k_1)\psi^*(k_2).\end{aligned}\tag{14}$$

This operator is essentially the same as the operator used in [11].

Theorem 1. *If g belongs to the Clifford group,*

$$\tau(x)=\langle\varphi(x)g\rangle\tag{15}$$

satisfies the bilinear equations of the KP hierarchy. The l-th modified τ function

$$\tau_{[k_1],\cdots[k_l]}(x)=e^{\xi(x,k_l)}e^{-\xi(\tilde{\partial},k_l^{-1})}\cdots e^{\xi(x,k_1)}e^{-\xi(\tilde{\partial},k_1^{-1})}\tau(x)\tag{16}$$

satisfies the following M^lKP equation (the l-th modified KP equation):

$$q_{l+1}(\tilde{D})\tau_{[k_1],\cdots,[k_l]}(x)\cdot\tau(x)=0.\tag{17}$$

We shall give below a sketch of the proof of Theorem 1. First we note that

$$\begin{aligned}\tau_{[k_1],\cdots,[k_l]}(x)=\langle[\cdots[[:e^{\rho(x)}\psi_0^*(-x)\cdots\psi_{l-1}^*(-x):,\\ \psi(k_l)k_l^{1-l}]_{(-)^l},\ \psi(k_{l-1})k_{l-1}^{2-l}]_{(-)^{l-1}},\ \cdots,\ \psi(k_1)]_+\rangle,\end{aligned}\tag{18}$$

$([\ ,\ ]_{(-)^l}=[\ ,\ ]_+(l:\text{ even}),\ =[\ ,\ ]\ (l:\text{ odd}))$.

Then the proof of (17) is done in the following manner. For simplicity, let us assume that $l=1$. Noting that

$$\chi_{\substack{\square\\\square}}(\tilde{D})f\cdot g=(\chi_{\substack{\square\\\square}}(\tilde{\partial})f)g-(\chi_{\square}(\tilde{\partial})f)(\chi_{\square}(\tilde{\partial})g)+f(\chi_{\square\square}(\tilde{\partial})g),$$

we have

$$\begin{aligned}&\chi_{\substack{\square\\\square}}(\tilde{D})\langle:e^{\rho(x)}\psi_0^*(-x):\psi(k)g\rangle\cdot\langle:e^{\rho(x)}:g\rangle\\ &\quad=\langle:e^{\rho(x)}\psi_{-1}(-x)\psi_0^*(-x)\psi_1^*(-x):\psi(k)g\rangle\langle:e^{\rho(x)}:g\rangle\\ &\qquad-\langle:e^{\rho(x)}\psi_1^*(-x):\psi(k)g\rangle\langle:e^{\rho(x)}\psi_{-1}(-x)\psi_0^*(-x):g\rangle\\ &\qquad+\langle:e^{\rho(x)}\psi_0^*(-x):\psi(k)g\rangle\langle:e^{\rho(x)}\psi_{-1}(x)\psi_1^*(-x):g\rangle\\ &\quad=0.\end{aligned}$$

In the last step we used Wick's theorem which is valid only if g belongs to the Clifford group. Noting (7) we can rewrite the bilinear equation (17) with $l=1$ as the linear equation (1) with $l=2$. The latter with $l\geq 3$ is obtained from (22) below with $l'=1$.

§ 3. For any Young diagram we set

$$h_Y^{(l)}(\tilde{D})=\det\begin{bmatrix}p_{f_1+l}(-\tilde{D}/2)p_{f_1+1}(\tilde{D}/2)\cdots p_{f_1+m-}1(\tilde{D}/2)\\ p_{f_2+l-1}(-\tilde{D}/2)p_{f_2}(\tilde{D}/2)\cdots p_{f_2+m-2}(\tilde{D}/2)\\ \vdots\qquad\qquad\vdots\qquad\qquad\vdots\\ p_{f_m+l-m+1}(-\tilde{D}/2)p_{f_m-m+2}(\tilde{D}/2)\cdots p_{f_m}(\tilde{D}/2)\end{bmatrix}.\tag{19}$$

Then we have

Theorem 2. *The hierarchy of the M^lKP ($l\geq 0$) equations contains the bilinear equation*

$$h_Y^{(l+1)}(\tilde{D})\tau_{[k_1],\dots,[k_l]}(x)\cdot\tau(x)=0 \tag{20}$$

for each Y.

We conjecture that the $h_Y^{(l+1)}(\tilde{D})$'s are linearly independent and exhaust all the bilinear equations of the $\mathrm{M}^l\mathrm{KP}$ hierarchy. Since the degree of $h_Y^{(l+1)}(\tilde{D})$ is $f_1+\cdots+f_m+l+1$, the number of degree n bilinear equations of the $\mathrm{M}^l\mathrm{KP}$ hierarchy is expected to be $p(n-l-1)$. The case $l=0$, i.e.

$$\begin{aligned}&\#\{\text{degree } n \text{ Hirota's bilinear equation of the KP hierarchy}\}\\&\quad=p(n-1),\end{aligned} \tag{21}$$

was conjectured by M. and Y. Sato in [8]. The proof of (21) will be given in one of our forthcoming papers [7], in which we discuss the relation between soliton equations and the Euclidean Lie algebras.

Now we shall give a sketch of the proof of Theorem 2. First we note the following [8].

Lemma. If $f_{[k]}(x)$ and $g(x)$ satisfy a bilinear equation $Q(D)f_{[k]}(x)\cdot g(x)=0$, then $f(x)$ and $g(x)$ satisfy

$$Q(D_1+k, D_2+k^2, \cdots)\exp(-(D_1/k+D_2/(2k^2)+\cdots)/2)f(x)\cdot g(x)=0.$$

Applying this lemma repeatedly to (17) we find that

$$Q(D\,;k_{l'+1},\cdots,k_l)\tau_{[k_1],\dots,[k_{l'}]}(x)\cdot\tau(x)=0$$

where

$$\begin{aligned}Q(D\,;k_{l'+1},\cdots,k_l)=&\frac{1}{k_{l'+1}\cdots k_l}q_{l+1}\Big(\tilde{D}_1+k_{l'+1}+\cdots+k_l,\\&\tilde{D}_2+\frac{k_{l'+1}^2}{2}+\cdots+\frac{k_l^2}{2},\cdots\Big)\\&\times\exp\Big(-\frac{1}{2}\Big\{\Big(\frac{1}{k_{l'+1}}+\cdots+\frac{1}{k_l}\Big)\tilde{D}_1+\Big(\frac{1}{k_{l'+1}^2}+\cdots+\frac{1}{k_l^2}\Big)\tilde{D}_2+\cdots\Big\}\Big).\end{aligned}$$

Setting $y_j=(k_{l'+1}^{-j}+\cdots+k_l^{-j})/2j$ and taking the limit $l\to\infty$ we obtain

$$\sum_{j=0}^{\infty}p_j(-2y)p_{j+l'+1}(\tilde{D}_X)\exp\Big(\sum_{j=1}^{\infty}y_jD_{x_j}\Big)\tau_{[k_1],\dots,[k_{l'}]}(x)\cdot\tau(x)=0. \tag{22}$$

Using Weyl's character formula we find the coefficient of $\chi_Y(y)$ in (22) to be $h_Y^{(l'+1)}(\tilde{D})$.

§4. In order to compute $\langle\varphi(x)g\rangle$ explicitly we fix the expectation value. The following choice of the vacuum was suggested by M. Sato.

$$\begin{aligned}&\psi_n|vac\rangle=0 \quad n\le -1,\ \langle vac|\psi_n=0 \quad n\ge 0,\\&\psi_n^*|vac\rangle=0 \quad n\ge 0,\ \langle vac|\psi_n^*=0 \quad n\le -1.\end{aligned} \tag{23}$$

We set

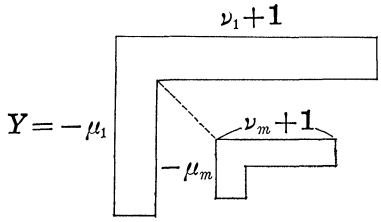
$$p_{\mu\nu}^-(x)=\sum_{l\le -1}p_{l-\mu}(-x)p_{\nu-l}(x)=\langle\psi_\nu(x)\psi_\mu^*(x)\rangle \qquad (\mu,\nu\in \mathbf{Z}). \tag{24}$$

Then we have

$$\rho(x+y)=\sum_{\mu,\nu\in\mathbf{Z}}\psi_\mu(-x)\psi_\nu^*(-x)p_{\mu\nu}^-(y). \tag{25}$$

If $\mu\le -1$ and $\nu\ge 0$, $(-)^{\mu-1}p_{\mu\nu}^-(x)$ is the character for $Y=(\nu+1, 1^{1-\mu})$ and

$\langle\psi_\mu^*(-x)\psi_\nu\rangle=\langle\psi_\nu(x)\psi_\mu^*\rangle=p_{-\mu+\nu}(x)$. Now let Y be a Young diagram of the following form:



The character $\chi_Y(x)$ is written as

$$(-)^{\mu_1+\cdots+\mu_m}\chi_Y(x)=\langle\varphi(x)\psi_{\mu 1}^*\cdots\psi_{\mu m}^*\psi_{\nu m}\cdots\psi_{\nu 1}\rangle=(-)^m\det\left(p^-_{\mu_j\nu_k}(x)\right)_{j,k=1,\cdots,m}.$$

Errata (Proc. Japan Acad., 56A, 9 (1980)). M. Jimbo and T. Miwa: Studies on Holonomic Quantum Fields. XVII.

p. 405, l. 9 from bottom: For "$(\sinh\beta_-E_1\sinh\beta_-E_2)^2=(\sinh\beta_+E_1\sinh\beta_+E_2)^{-2}$" read "$(\sinh 2\beta_-E_1\sinh 2\beta_-E_2)^2=(\sinh 2\beta_+E_1\sinh 2\beta_+E_2)^{-2}$." l. 2 from bottom: For

$$\text{“}N^2\left((t-1)\frac{d\sigma}{dt}-\sigma\right)^2-\frac{d\sigma}{dt}\left((t-1)\frac{d\sigma}{dt}-\sigma-\frac{1}{2}\right)\left((t+1)\frac{d\sigma}{dt}-\sigma\right)\text{”}$$

read

$$\text{“}N^2\left((t-1)\frac{d\sigma}{dt}-\sigma\right)^2-4\frac{d\sigma}{dt}\left((t-1)\frac{d\sigma}{dt}-\sigma-\frac{1}{4}\right)\left(t\frac{d\sigma}{dt}-\sigma\right).\text{”}$$

p. 406, l. 20: For "$\beta_0^{(-)}=\alpha_0\sqrt{t-1}(F(-1/2,1/2,1;1/t)-F(1/2,1/2,1;1/t))$" read "$\beta_0^{(-)}=-\alpha_0\sqrt{t}^{-1}\sqrt{t-1}(F(-1/2,1/2,1;1/t)-F(1/2,1/2,1;1/t))$." l. 11 from bottom: For "$(\sinh\beta_-E_1\sinh\beta_-E_2)^{-1}$" read "$(\sinh 2\beta_-E_1\sinh 2\beta_-E_2)^{-1}$."

p. 407, l. 2: For "$\sinh\beta_+E_1\sinh\beta_+E_2$" read "$\sinh 2\beta_+E_1\sinh 2\beta_+E_2$."

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