7. *Scattering Techniques in Transmutation and some Connection Formulas for Special Functions*

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1. Introduction. Fadeev in [11] develops a technique for displaying certain operators of interest in scattering theory in terms of transmutations; this allows one to give an essentially unified derivation of the Gelfand-Levitan and Marčenko equations (which is generalized in Carroll [6]). In particular the link between the Gelfand-Levitan and Marčenko equations is shown in [11] to be a certain transmutation operator $\tilde{U}$ and in this article we determine the natural generalization $\tilde{\mathcal{B}}$ (or $\tilde{\mathcal{B}}$) of $\tilde{U}$ in the transmutation framework of Carroll [2]–[5]; then, in a context based on harmonic analysis in rank one noncompact symmetric spaces, we show how the use of such operators $\tilde{\mathcal{B}}$ provides a transmutation meaning and abstract derivation for various types of formulas connecting special functions with integrals of Riemann-Liouville and Weyl type (cf. Flensted-Jensen [12], Koornwinder [13], Askey-Fitch [1], Chao [8]). One particular feature of $\tilde{U}$ which relates Riemann-Liouville and Weyl type integrals in the relation $\tilde{U}=(U^{-1})^*$ for a basic transmutation operator U and this provides complementary types of triangular kernels (cf. here Erdélyi [10] for a related use of adjointness). In our more general framework adjointness plays a different role but we obtain similar triangularity results for the analogous $\mathcal{B}$ and $\tilde{\mathcal{B}}$ by other methods (Theorem 2.1). The details will appear in [7].

2. Basic constructions. We will work with differential operators of the form $P(D)u=(Au')'/A$ where $A(x)$ will have properties modeled on $P(D)$ being the radial Laplace-Beltrami operator on a noncompact Riemannian symmetric space of rank one (cf. [9], [12], [13] for details). Let $\varphi_\lambda^P(t)$ be a "spherical function" satisfying $P(D)\varphi_\lambda^P=(-\lambda^2-\rho^2)\varphi_\lambda^P$, $\varphi_\lambda^P(0)=1$, and $D_t\varphi_\lambda^P(0)=0$, where $\rho=\lim(1/2)A'/A$ at $t\to\infty$. Thus $\varphi_\lambda(t)=\varphi_\lambda^P(t)\sim H(t,\mu)$ for $\mu=-\lambda^2$ and $\hat{P}=P+\rho^2$ (notation of [2]–[5]). We set $\Omega(x,\mu)=\Omega_\lambda(x)=\Omega_\lambda^P(x)=\Delta_P(x)\varphi_\lambda^P(x)$ where $\Delta_P(x)=A(x)$ for $P(D)$. Then $\hat{P}^*(D)\Omega_\lambda^P=\mu\Omega_\lambda^P$ where $P^*(D)\psi=[A(\psi/A)']'$ denotes the formal adjoint of $P(D)$. A typical example of $\Delta_P(x)$ here is $\Delta_P(x)=\Delta_{\alpha\beta}(x)=(e^x-e^{-x})^{2\alpha+1}(e^x+e^{-x})^{2\beta+1}$ with $\rho=\alpha+\beta+1$ in which

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case the spherical functions $\varphi_\lambda^P(x)$ are Jacobi functions of the first kind $\varphi_\lambda^{\alpha\beta}(x)=F(2^{-1}(\rho+i\lambda),\ 2^{-1}(\rho-i\lambda),\ \alpha+1,\ -sh^2x)$ (cf. [13]). A second solution of $\hat{P}(D)\psi=\mu\psi$ in this situation is given by the function $\Phi_\lambda^{\alpha\beta}(x)=\Phi_\lambda^P(x)=(e^x-e^{-x})^{i\lambda-\rho}F(2^{-1}(\beta-\alpha+1-i\lambda),\ 2^{-1}(\beta+\alpha+1-i\lambda),\ 1-i\lambda,\ -sh^{-2}x)$ which is called a Jacobi function of the second kind and which we shall refer to as a Jost solution (cf. [7], [11]). Indeed one has $\Phi_\lambda^P(x)\sim\exp(i\lambda-\rho)x$ as $x\to\infty$ and $\varphi_\lambda(x)=c(\lambda)\Phi_\lambda(x)+c(-\lambda)\Phi_{-\lambda}(x)$ where $c(\lambda)=c_P(\lambda)$ is the Harish-Chandra function (which corresponds essentially here to the Jost function of physics). A related example in [12] involves $\Delta_P(x)=\Delta^{p,q}(x)=(e^x-e^{-x})^p(e^{2x}-e^{-2x})^q$. Analyticity and growth properties of φ_λ and Φ_λ can be found in [12], [13].

We will assume our operators $P(D)$ are of a type where $A(x)\sim\Delta_{\alpha\beta}(x)$ or $\Delta^{p,q}(x)$ and suitable analyticity and growth properties are valid (cf. also [9]). Now recall the notation of [2], [4], [5] which we modify slightly in writing

$$\hat{f}(\lambda)=\mathfrak{P}f(\lambda)=\int_0^\infty f(x)\varphi_\lambda^P(x)\Delta_P(x)dx$$

and

$$f(x)=\mathsf{P}\hat{f}(x)=\int_0^\infty \hat{f}(\lambda)\varphi_\lambda^P(x)d\nu_P(\lambda)$$

where $d\nu(\lambda)=d\nu_P(\lambda)=d\lambda/2\pi|c_P(\lambda)|^2$ (we will write $\mathfrak{P}f(\lambda)=\langle f(x),\ \Omega_\lambda^P(x)\rangle$ and $\mathsf{P}\hat{f}(x)=\langle\hat{f}(\lambda),\ \varphi_\lambda^P(x)\rangle_\nu$). Similar transformations are defined relative to another operator $Q(D)$ as above in the form

$$\tilde{g}(\lambda)=\mathfrak{Q}g(\lambda)=\int_0^\infty g(x)\varphi_\lambda^Q(x)\Delta_Q(x)dx \quad \text{with} \quad \mathsf{Q}=\mathfrak{Q}^{-1};$$

we will write $d\omega_Q(\lambda)=d\omega(\lambda)=d\lambda/2\pi|c_Q(\lambda)|^2$. Let us also define

$$\hat{\hat{h}}(\lambda)=\mathscr{P}h(\lambda)=\int_0^\infty h(x)\varphi_\lambda^P(x)dx,\ \boldsymbol{P}\hat{\hat{h}}(x)=\mathscr{P}^{-1}\hat{\hat{h}}(x)=\int_0^\infty\hat{\hat{h}}(\lambda)\varphi_\lambda^P(x)\Delta_P(x)d\nu,$$

with corresponding maps $\mathcal{Q}$ and $\boldsymbol{Q}=\mathcal{Q}^{-1}$, while we set $\Pi F(x)=\langle F(\lambda),\ \varphi_\lambda^P(x)\rangle_\omega$ and $\Xi G(x)=\langle G(\lambda),\ \varphi_\lambda^Q(x)\rangle_\nu$. Note that

$$\delta_P(x)=\delta(x)/\Delta_P(x)=\int_0^\infty\varphi_\lambda^P(x)d\nu$$

with $\hat{\delta}_P(\lambda)=1$. A framework of spaces and maps is developed in [2], [4], [5] and we refer to [7] for details. Transmutation operators B and $\mathscr{B}=B^{-1}$ satisfying $B\hat{P}=\hat{Q}B$ and $\mathscr{B}\hat{Q}=\hat{P}\mathscr{B}$ are constructed in the form $B=\Xi\mathfrak{P}$ and $\mathscr{B}=\Pi\mathfrak{Q}$ where $B^*=\boldsymbol{P}\mathcal{Q}$, $\mathscr{B}^*=\boldsymbol{Q}\mathscr{P}$, and $\Xi^{-1}=\mathfrak{P}\Pi\mathfrak{Q}$; one says $B:\hat{P}\to\hat{Q}$ and $\mathscr{B}:\hat{Q}\to\hat{P}$ where we have set $\hat{P}u=Pu+\rho_P^2u$ and $\hat{Q}u=Qu+\rho_Q^2u$. The operators B and $\mathscr{B}$ have kernel expressions $Bf(y)=\langle\beta(y,x),\ f(x)\rangle$ and $\mathscr{B}g(x)=\langle\gamma(x,y),\ g(y)\rangle$ where $\beta(y,x)=\langle\Omega_\lambda^P(x),\ \varphi_\lambda^Q(y)\rangle_\nu$ and $\gamma(x,y)=\langle\varphi_\lambda^P(x),\ \Omega_\lambda^Q(y)\rangle_\omega$.

Let now $W(\lambda)=|c_Q(\lambda)/c_P(\lambda)|^2$ so that $d\nu_P=W(\lambda)d\omega_Q$. One knows that $\varphi_\lambda^P=\mathscr{B}\varphi_\lambda^Q$ and one defines now $\tilde{\mathscr{B}}=\mathsf{P}\mathfrak{Q}$ so that $W(\lambda)\varphi_\lambda^P=\tilde{\mathscr{B}}\varphi_\lambda^Q$ (which follows the spirit of [11]). Then setting $W^x=\mathsf{Q}W(\lambda)\mathfrak{Q}$, we have

Theorem 2.1. *$\tilde{\mathcal{B}}=\mathsf{P}\mathfrak{Q}$ is a transmutation $\tilde{\mathcal{B}}\hat{Q}=\hat{P}\tilde{\mathcal{B}}$, $W(\lambda)\varphi_\lambda^P=\tilde{\mathcal{B}}\varphi_\lambda^Q$, $\tilde{\mathcal{B}}=\mathcal{B}W^x$, $\tilde{\mathcal{B}}g(x)=\langle\tilde{\gamma}(x,y), g(y)\rangle$ where $\tilde{\gamma}(x,y)=\langle\varphi_\lambda^P(x), \Omega_\lambda^Q(y)\rangle_\nu=\Delta_Q(y)\Delta_P^{-1}(x)\beta(y,x)$, $\gamma(x,\cdot)\in\mathcal{E}'_y$ with $\gamma(x,y)=0$ for $y>x$, and $\tilde{\gamma}(\cdot,y)\Delta_P(\cdot)\Delta_Q^{-1}(y)=\beta(y,\cdot)\in\mathcal{E}'_x$ with $\tilde{\gamma}(x,y)=0$ for $x>y$.*

The triangularity proof involves writing $\varphi_\lambda^P(y)=\mathcal{B}\varphi_\lambda^Q(y)=\Pi\mathfrak{Q}\varphi_\lambda^Q(y)=Q\gamma(y,\cdot)(\lambda)=\mathfrak{Q}[\gamma(y,\cdot)/\Delta_Q(\cdot)](\lambda)$. Similarly from $W(\lambda)\varphi_\lambda^P(x)=\tilde{\mathcal{B}}\varphi_\lambda^Q(x)$ with $\tilde{\mathcal{B}}=\mathsf{P}\mathfrak{Q}$ we get $\tilde{\gamma}(x,y)/\Delta_Q(y)=\mathsf{Q}[W(\lambda)\varphi_\lambda^P(x)](y)=\mathsf{P}[\varphi_\lambda^Q(y)](x)$ so that $\varphi_\lambda^Q(y)=\mathfrak{P}[\tilde{\gamma}(\cdot,y)/\Delta_Q(y)](\lambda)$. Then the Paley-Wiener theorem can be used.

In the case where $P\sim\Delta_{\alpha\beta}$ and $Q\sim\Delta_{\alpha+\mu,\beta+\mu}$ some formulas in [13] based on known relations between hypergeometric functions can be recast to produce

Theorem 2.2. *For $P\sim\Delta_{\alpha\beta}$ and $Q\sim\Delta_{\alpha+\mu,\beta+\mu}$ one has*

$$\tilde{\mathcal{B}}\left(\frac{\Phi_\lambda^Q(y)}{c_Q(-\lambda)}\right)=\frac{\Phi_\lambda^P(x)}{c_P(-\lambda)}. \tag{2.1}$$

3. Connection formulas. For various reasons (mainly to avoid distribution kernels) we take now $P=D^2$ and $Q\sim\Delta_Q$ as before (instead of $Q=D^2$ as in [5] or [11]). Thus $\varphi_\lambda^P(t)=\operatorname{Cos}\lambda t$, $\Phi_\lambda^P(t)=e^{i\lambda t}$, $\Delta_P=1$, and $c_P(\lambda)=1/2$. We will write kernels for this situation as $\beta_Q(y,x)$, $\gamma_Q(x,y)$, etc. First using complex variable arguments modeled on [11] (with no recourse to properties of hypergeometric functions) one proves a direct generalization of a formula of [11] in the form

Theorem 3.1. *For $Q\sim\Delta_Q$ we have*

$$\frac{e^{i\lambda x}}{1/2}=\tilde{\mathcal{B}}\left(\frac{\Phi_\lambda^Q(y)}{c_Q(-\lambda)}\right)(x). \tag{3.1}$$

This is a special case of Theorem 2.2 but the demonstration is "abstract". A (different) abstract proof of Theorem 2.2 can also be produced. Further in this context it is natural to utilize the operator $\hat{\mathcal{B}}=\mathsf{Q}\mathfrak{P}=\tilde{\mathcal{B}}^{-1}$ so that $\hat{\mathcal{B}}\mathcal{B}W^x=I$, $\mathcal{B}^*=\Delta_Q(y)\hat{\mathcal{B}}$, and $\hat{\mathcal{B}}f(y)=\langle\hat{\beta}_Q(y,x), f(x)\rangle$ with $\hat{\beta}_Q(y,x)=\langle\varphi_\lambda^Q(y), \operatorname{Cos}\lambda x\rangle_\omega=0$ for $y>x$.

Note that $\hat{\mathcal{B}}=\mathsf{Q}\mathfrak{P}$ is defined quite generally; note also that since we have reversed the position of D^2 from [11] it is $\hat{\mathcal{B}}$ which corresponds to $\tilde{U}$ here. Thus (3.1) holds and $\check{\gamma}_Q(x,y)=\Delta_Q(y)\beta_Q(y,x)$. From [4], [5], [14] we now know $\mathcal{P}f=Q\check{f}$ for $\check{f}=\mathcal{B}^*f$ and $\mathcal{P}\check{g}=Qg$ for $\check{g}=B^*g$. In the present context we have $B^*[\Delta_Q f]=\hat{\mathcal{B}}f$ and $\mathcal{P}B^*[\Delta_Q f](x)=Q[\Delta_Q f](x)=\mathfrak{Q}f(x)$. Hence $(Q\sim\Delta_{\alpha\beta})$ and, referring to [13] for F_Q, we obtain

Theorem 3.2. *$F_Q[f](x)=B^*[\Delta_Q f](x)$ and $\mathcal{P}F_Q[f]=\mathfrak{Q}f$.*

Another set of formulas in [13] use Weyl type integrals W_μ^σ. We can represent $W_{\beta+1/2}^2$ as a transmutation $W_{\beta+1/2}^2=\Gamma(\alpha+1)\tilde{\mathcal{B}}/2^{3\beta+3/2}\Gamma(\alpha+\beta+1/2)$ where, in an obvious notation, $\tilde{\mathcal{B}}:(\alpha,\beta)\to(\alpha-\beta-1/2, -1/2)$. Similarly $W_{\alpha-\beta}^1=\sqrt{\pi}\tilde{\mathcal{B}}'/2^{3(\alpha-\beta)}\Gamma(\alpha-\beta+1/2)$ where $\tilde{\mathcal{B}}':(\alpha-\beta-1/2, -1/2)\to(-1/2, -1/2)$. Then for $\tilde{\mathcal{B}}_Q:(\alpha,\beta)\to(-1/2, -1/2)$ as in Theorem

3.2 (i.e. $\tilde{\mathscr{B}}_Q f = B^*[\varDelta_Q f]$, $(-1/2, -1/2) \sim D^2$, $(\alpha, \beta) \sim Q$) the formula $F_{\alpha\beta} = 2^{3\alpha+3/2} W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}$ of [13] is equivalent to

Theorem 3.3. *The operator $F_Q[f] = \tilde{\mathscr{B}}_Q f$ can be factored as*

$$F_Q = \frac{\sqrt{\pi}\,\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1/2)\Gamma(\alpha+\beta+3/2)} \tilde{\mathscr{B}}' \circ \tilde{\mathscr{B}} \tag{3.2}$$

for $\tilde{\mathscr{B}}$ and $\tilde{\mathscr{B}}'$ as indicated.

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