# 17. *Some Prehomogeneous Vector Spaces with Relative Invariants of Degree Four and the Formula of the Fourier Transforms*

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(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 12, 1980)

In this article, we shall investigate the relative invariant $f(x)$ of a regular prehomogeneous vector space $(G, V)$ when it is one of the following ones; 1) $SL(6)\times GL(1)$ $(\Lambda_3\times\Lambda_1)$, 2) $Sp(3)\times GL(1)$ $(\Lambda_3\times\Lambda_1)$, 3) $Spin(12)\times GL(1)$ ((half-spin rep.)$\times\Lambda_1$), 4) $E_7\times GL(1)$ ((56 dim. rep.)$\times\Lambda_1$), where $\Lambda_i$ is the representation on the space of the skew-symmetric tensors of rank $i$. The polynomial $f(x)$ has the following form,

(1) $$f(x)=(x_0y_0-\langle X, Y\rangle)^2+4x_0N(Y)+4y_0N(X)-4\langle X^\#, Y^\#\rangle.$$

Here, $x=(x_0, y_0, X, Y)\in C\oplus C\oplus C^m\oplus C^m$ and $\langle X, Y\rangle$ is some bilinear form in $X$ and $Y$, $N(X)$ is some polynomials in $X$, and $X\mapsto X^\#$ is some polynomial mapping from the $X$-space into itself.

We shall calculate the Fourier transform of the hyperfunction $|f(x)|^s$ for a generic $s\in C$. As shown in [5], the formula of the Fourier transform gives the functional equation of the local zeta function associated with the prehomogeneous vector spaces.

1. Let $u_1, \cdots, u_6$ be a basis of the six-dimensional complex vector space $E$ with the natural action of $G=SL(6)\times GL(1)$, i.e., $(u_1, \cdots, u_6)\mapsto C_2(u_1, \cdots, u_6){}^tg_1$ for $(g_1, c)\in SL(6)\times GL(1)$. We denote by $V(20)$ the vector space of the skew-symmetric tensors on $E$ of rank 3 and $x_{ijk}$ denotes the coefficient of $u_i\wedge u_j\wedge u_k$. The complex algebraic group $SL(6)\times GL(1)$ acts on $V(20)$, and it is a regular prehomogeneous vector space. We identify $V(20)$ and $C\oplus C\oplus M(3, C)\oplus M(3, C)$ by

(2) $$x_0=x_{123} \qquad y_0=x_{456}$$
$$X=\begin{pmatrix} x_{423}, & x_{143}, & x_{124} \\ x_{523}, & x_{153}, & x_{125} \\ x_{623}, & x_{163}, & x_{126}\end{pmatrix} \qquad Y=\begin{pmatrix} x_{156}, & x_{416}, & x_{451} \\ x_{256}, & x_{426}, & x_{452} \\ x_{356}, & x_{436}, & x_{453}\end{pmatrix}.$$

By setting $\langle X, Y\rangle=\mathrm{tr}\,(X\cdot Y)$, $N(X)=\det X$, and $X^\#=$ the cofactor matrix of $X$, $f(x)$ is an irreducible relatively invariant polynomial on the prehomogeneous vector space $(G, V)=(SL(6)\times GL(1), V(20))$ with the character $\chi(g_1, c)=c^{12}$. This is the prehomogeneous vector space 1). We define the symplectic group $Sp(3)$ as the subgroup of $SL(6)$ consisting of the elements which leave $u_1\wedge u_4+u_2\wedge u_5+u_3\wedge u_6$ invariant. When we set

(3) $$V(14)=\{(x_0, y_0, X, Y)\in V(20)\,;\ {}^tX=X,\ {}^tY=Y\},$$

$V(14)$ is an invariant subspace under the actions of $Sp(3)\times GL(1)$, and $(G, V)=(Sp(3)\times GL(1), V(14))$ is a regular prehomogeneous vector space. The restriction of $f(x)$ on $V(14)$ is a relative invariant corresponding to the character $\chi(g_1, C)=C^{12}$. This is the prehomogeneous vector space 2).

Next consider the even half-spin representation of the complex spinor group $Spin(12)$. We denote by $V(32)$ the space of skew-symmetric tensors of even rank on the six dimensional complex vector space $E$ and let $\{e_1, \cdots, e_6\}$ be a basis of $E$. We denote an element of $V(32)$ by

$$x=x_0+\sum_{i<j} x_{ij}e_ie_j+\sum_{i<j} y_{ij}e^*_{ij}+y_0e_L, \tag{4}$$

where $e_L=e_1e_2e_3e_4e_5e_6$ and $e^*_{ij}$ is the element of the form $e_ke_le_me_n$ satisfying $e_ie_je^*_{ij}=e_L$, and $X, Y$ denote the $6\times 6$ skew-symmetric matrices whose $i$-$j$ entries are $x_{ij}$ and $y_{ij}$ for $i<j$, respectively. Then $Spin(12)$ acts on $V(32)$ as the even half-spin representation (see J. Igusa [1]), and $(G, V)=(Spin(12)\times GL(1), V(32))$ is a regular prehomogeneous vector space. Here $GL(1)$ acts on $V(32)$ by the multiplication. The polynomial $f(x)$ is an irreducible relative invariant by setting $\langle X, Y\rangle = -\mathrm{tr}\,(X\cdot Y)/2$, $N(X)=\mathrm{Pff}\,(X)$ and $X^\#$ is the $6\times 6$ skew-symmetric matrix whose $i$-$j$ entry is $\pm\mathrm{Pff}\,(X_{ij})$ for $i\leqslant j$, where $X_{ij}$ is the $4\times 4$ skew-symmetric matrix obtained by crossing out the $i$-th and $j$-th columns and rows. The character of $f(x)$ is $\chi(g_1, c)=c^4$ for $(g_1, c)\in Spin(12)\times GL(1)$. This is the prehomogeneous vector space 3). Here, $\mathrm{Pff}(X)$ is the Pffafian of $X$ normalized by

$$\mathrm{Pff}\begin{bmatrix} -1^{1} & & & \\ & -1^{1} & & \\ & & \ddots & \\ & & & -1^{1} \end{bmatrix}=1.$$

Next we shall consider the exceptional complex algebraic group $E_7$ and the 56-dimensional representation of $E_7$. The representation space $V(56)$ is

$$\{(x_0, y_0, X, Y)\,;\ x_0, y_0\in C \text{ and } X, Y\in \mathcal{J}\}, \tag{5}$$

where $\mathcal{J}$ is the exceptional simple Jordan algebra over $C$ (see N. Jacobson [2]). An element $X$ of $\mathcal{J}$ is denoted by

$$X=\begin{pmatrix} \xi_1, \bar{x}_3, x_2 \\ x_3, \xi_2, \bar{x}_1 \\ \bar{x}_2, x_1, \xi_3 \end{pmatrix} \qquad \begin{matrix} \xi_1, \xi_2, \xi_3\in C \\ x_1, x_2, x_3\in \mathfrak{L}, \end{matrix} \tag{6}$$

where $\mathfrak{L}$ is the Cayley algebra over $C$. We define the norm of $X$ by $\det X=\xi_1\xi_2\xi_3+\mathrm{tr}\,(x_1x_2x_3)-\sum \xi_i x_i\bar{x}_i$ and the trace of $X$ by $\mathrm{tr}\,(X)=\xi_1+\xi_2+\xi_3$. We set $S(X)=(\mathrm{tr}\,(X)^2-\mathrm{tr}\,(X^2))/2$. Then $(G, V)=(E_7\times GL(1), (V56))$ is a regular prehomogeneous vector space. Here $GL(1)$ acts on $V(56)$ by the multiplication. The polynomial $f(x)$ is an irreducible relative invariant by setting $\langle X, Y\rangle=\operatorname{tr}(XY+YX)/2$, $X^{\#}=X^2-\operatorname{tr}(X)\cdot X+S(X)\cdot I$, and $N(X)=\det X$. The character of $f(x)$ is $\chi(g_1, c)=c^4$ for $(g_1, c)\in \boldsymbol{E}_7\times \boldsymbol{GL}(1)$. This is the prehomogeneous vector space 4).

**2.** The $b$-function of $f^s(x)$ is calculated by micro-local calculus (see T. Kimura [3]), and it is

$$b(s)=(s+1)\left(s+\frac{l+3}{2}\right)\left(s+\frac{2l+3}{2}\right)\left(s+\frac{3l+4}{2}\right), \tag{7}$$

for 1) $l=2$, 2) $l=1$, 3) $l=4$, 4) $l=8$, respectively.

**3.** The prehomogeneous vector spaces 1)–4) have the following real forms $(G_R, V_R)$.

1) 1)–i) $G_R=SU(3,3,C)\times GL(1,R)$

$$V_R=\left\{\begin{matrix}(x_0\cdot\sqrt{-1}y_0, X, Y)\,;\ x_0, y_0\in R.\ \ X, Y\in M(3,C) \\ \text{and}\ \ {}^tX=-X,\ {}^tY=Y\end{matrix}\right\}.$$

1)–ii) $G_R=SL(6,R)\times GL(1,R)$

$$V_R=\{(x_0, y_0, X, Y)\,;\ x_0, y_0\in R.\ \ X, Y\in M(3,R)\}.$$

1)–iii) $G_R=SU((1,5,C)\times GL(1,R)$

$$V_R=\left\{\left(x_0, -\bar{x}_0, X, {}^t\overline{X}\begin{pmatrix} & & -1\\ & 1 & \\ 1 & & \end{pmatrix}\right)\,;\ x_0\in C,\ X\in M(3,C)\right\}.$$

2) 2)–i) $G_R=Sp(3,R)\times GL(1,R)$

$$V_R=\left\{\begin{matrix}(x_0, y_0, X, Y)\,;\ x_0, y_0\in R.\ X, Y\in M(3,R) \\ \text{and}\ \ {}^tX=X,\ {}^tY=Y\end{matrix}\right\}.$$

3) 3)–i) $G_R=Spin(6,H)\times GL(1,R)$

$$V_R=\{(x_0, \bar{x}_0, X, \overline{X})\,;\ x_0\in C.\ X\in M(6,C)\ \text{and}\ {}^tX=-X\}.$$

3)–ii) $G_R=Spin(6,6,R)\times GL(1,R)$

$$V_R=\left\{\begin{matrix}(x_0, y_0, X, Y)\,;\ x_0, y_0\in R.\ \ X, Y\in(6,R) \\ \text{and}\ \ {}^tX=-X,\ {}^tY=-Y\end{matrix}\right\}.$$

3)–iii) $G_R=Spin(10,2,R)\times GL(1,R)$

$$V_R=\left\{\begin{matrix}(x_0, y_0, X, Y)\,;\ x_0, y_0\in C. \\ X=\left(\begin{array}{c|c} X_1 & X_2 \\ \hline -{}^tX_2 & \sqrt{-1}\tilde{y}_0\end{array}\right),\ \ Y=\left(\begin{array}{c|c} Y_1 & \tilde{X}_2 \\ \hline -{}^t\tilde{X}_2 & \sqrt{-1}\tilde{x}_0\end{array}\right)\end{matrix}\right\},$$

where $X_2\in M(4,2,C)$, $\tilde{X}_2=\sqrt{-1}\overline{X}_2\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$, $\tilde{x}_0=\bar{x}_0\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$, $\tilde{y}_0=\bar{y}_0\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$,

$$X_1=\begin{bmatrix} 0, & x_2, & x_3, & x_4 \\ -x_2, & 0, & -\sqrt{-1}\bar{x}_4, & \sqrt{-1}\bar{x}_3 \\ -x_3, & \sqrt{-1}\bar{x}_4, & 0, & -\sqrt{-1}\bar{x}_2 \\ -x_4, & -\sqrt{-1}\bar{x}_3, & \sqrt{-1}\bar{x}_2, & 0\end{bmatrix}\ \text{and}$$

$$Y_1=\begin{bmatrix} 0, & -y_2, & -y_3, & -y_4 \\ y_2, & 0, & \sqrt{-1}\bar{y}_4, & -\sqrt{-1}\bar{y}_3 \\ y_3, & -\sqrt{-1}\bar{y}_4, & 0, & \sqrt{-1}\bar{y}_2 \\ y_4, & \sqrt{-1}\bar{y}_3, & -\sqrt{-1}\bar{y}_2, & 0\end{bmatrix},\ \text{with}\ x_i, y_i\in C.$$

4) 4)–i) $G_R=E_7^d\times GL(1,R)$

$$V_R=\{(x_0, y_0, X, Y)\,;\ x_0, y_0\in R.\ X, Y\in \mathcal{J}^d\}.$$

4)–ii) $G_R=E_7^s\times GL(1,R)$

$$V_{\boldsymbol{R}}=\{(x_0, y_0, X, Y)\,;\, x_0, y_0 \in \boldsymbol{R}.\ X, Y \in \mathcal{J}^s\}.$$

Here, $\boldsymbol{E}_7^d$ and $\boldsymbol{E}_7^s$ are real forms of $E_7$ whose Killing forms have the signature $-25$ and 7, respectively, and $\mathcal{J}^d$ and $\mathcal{J}^s$ are the spaces of $3\times 3$ octanion Hermitian matrices whose entries are Cayley division numbers and split Cayley numbers over $\boldsymbol{R}$, respectively.

4. We define the inner product on $V$ by

$$\langle x, x'\rangle = x_0 y_0' - x_0' y_0 - \langle X, Y'\rangle + \langle X', Y\rangle.$$

We define the real-valued inner product on $V_{\boldsymbol{R}}$ by restricting this on $V_{\boldsymbol{R}}$ and by multiplying a constant of absolute value one if necessary. We denote by $dx$ the Euclidian measure on $V_{\boldsymbol{R}}$ satisfying

$$(2\pi)^n u(x'') = \iint u(x)\exp(\sqrt{-1}\langle x, x'\rangle)\exp(-\sqrt{-1}\langle x', x''\rangle)\,dx\,dx',$$

where $n=\dim V_{\boldsymbol{R}}$.

The open set $V_{\boldsymbol{R}}-\{f=0\}$ decomposes into the following three connected components, which are $\boldsymbol{G}_{\boldsymbol{R}}^+$-orbits.

$$(8)\quad V_1=\boldsymbol{G}_{\boldsymbol{R}}^+\cdot\left(1, 0, (0), \begin{pmatrix}1 & & \\ & -1 & \\ & & -1\end{pmatrix}\right)$$

$$\left(\text{resp. } \boldsymbol{G}_{\boldsymbol{R}}^+\left(1, 0, (0), \begin{pmatrix}1 & & \\ & 1 & \\ & & -1\end{pmatrix}\right);\ \boldsymbol{G}_{\boldsymbol{R}}^+\cdot\left(0, 0, \left(-1^{1}{}_{0_{0}}\right), \left(-1^{1}{}_{0_{0}}\right)\right)\right)$$

$$V_2=\boldsymbol{G}_{\boldsymbol{R}}^+\cdot\left(1, 0, (0), \begin{pmatrix}1 & & \\ & 1 & \\ & & -1\end{pmatrix}\right)$$

$$\left(\text{resp. } \boldsymbol{G}_{\boldsymbol{R}}^+\cdot\left(1, 0, (0), \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}\right);\right.$$

$$\left.\boldsymbol{G}_{\boldsymbol{R}}^+\cdot\left(1, 1, \begin{bmatrix}-1^{1} & & \\ & -1^{1} & \\ & & -1^{1}\end{bmatrix}, \begin{bmatrix}-1^{1} & & \\ & -1^{1} & \\ & & {}_{1}{-1}\end{bmatrix}\right)\right)$$

$$V_3=\boldsymbol{G}_{\boldsymbol{R}}^+\cdot\left(1, 0, (0), \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}\right)$$

$$\left(\text{resp. } \boldsymbol{G}_{\boldsymbol{R}}^+\cdot\left(-1, 0, (0), \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}\right);\ \boldsymbol{G}_{\boldsymbol{R}}^+\cdot(1, 1, (0), (0))\right)$$

in the case of 1)–i) (resp. 2)–i) and 4)–i); 3)–i)).

The hyperfunction

$$(9)\qquad |f|_i^s(x)=\begin{cases}|f(x)|^s & x\in V_i\\ 0 & x\notin V_i\end{cases}$$

is defined first for $\mathrm{Re}(s)\gg 0$ and continued to $\boldsymbol{C}$ meromorphically. By identifying $V_{\boldsymbol{R}}$ and $V_{\boldsymbol{R}}^*$ by the inner product $\langle x, x'\rangle$, $|f|_i^s(x')$ is defined on $V_{\boldsymbol{R}}^*$. The Fourier transform of $|f|_i^s(x)$ is the following:

$$(10)\quad \int\begin{bmatrix}|f|_1^s(x)\\ |f|_2^s(x)\\ |f|_3^s(x)\end{bmatrix}\cdot\exp(\sqrt{-1}\langle x, x'\rangle)\,dx$$

$$=(2\pi)^{3l+2}\cdot\Gamma(s+1)\Gamma\left(s+\frac{l+3}{2}\right)\Gamma\left(s+\frac{2l+3}{2}\right)\Gamma\left(s+\frac{3l+4}{2}\right)\cdot 4^{2s+n/4}$$

$$\cdot\begin{bmatrix}(-1)^l\cdot 2\cdot\sin(2\pi s), & (2+(-1)^l)\cdot 2\cdot\cos(\pi s-(\pi(l-1)/2)), & 0\\ 0, & (\sqrt{-1})^{l-1}+(-\sqrt{-1})^{l-1}+2\cos(2\pi s-(\pi(l-1)/2)), & 0\\ 0, & 2\cdot\cos(\pi s-(\pi(l-1)/2)), & (-1)^l\cdot 2\cdot\sin(2\pi s)\end{bmatrix}\cdot\begin{bmatrix}|f|_1^{-s-(n/4)}(x')\\ |f|_2^{-s-(n/4)}(x')\\ |f|_3^{-s-(n/4)}(x')\end{bmatrix},$$

for $l=2, 1, 4$ and $8$ in the case of 1)–$i$), 2)–i), 3)–i) and 4)–i), respectively.

In the case of 1)–ii), 3)–ii), and 4)–ii), the open set $V_R-\{f=0\}$ decomposes into two connected components $V_\pm=\{f(x)\gtrless 0\}$. We can define $|f|_\pm^s(x)$ in the same way as (9) for a generic $s\in C$. The Fourier transform of $|f|_\pm^s(x)$ is as follows:

$$(11)\quad \int\begin{bmatrix}|f|_+^s(x)\\ |f|_-^s(x)\end{bmatrix}\exp(-\sqrt{-1}\langle x, x'\rangle)dx$$
$$=(2\pi)^{3l+2}\cdot\Gamma(s+1)\Gamma\left(s+\frac{l+3}{2}\right)\Gamma\left(s+\frac{2l+3}{2}\right)\Gamma\left(s+\frac{3l+4}{2}\right)\cdot 4^{2s+n/4}$$
$$\cdot\begin{bmatrix}(-1)^{l/2}\cdot 2\cdot\sin(-2\pi s), & 0\\ (1+(\sqrt{-1})^l+(\sqrt{-1})^{2l}+(\sqrt{-1})^{3l})\cdot 2\cdot\sin(\pi s), & (-1)^{l/2}\sin(2\pi s)\end{bmatrix}$$
$$\cdot\begin{bmatrix}|f|_+^{-s-(n/4)}(x')\\ |f|_-^{-s-(n/4)}(x')\end{bmatrix}$$

for $l=2, 4$ and $8$ in the case of 1)–ii), 3)–ii) and 4)–ii), respectively.

In the case of 1)–iii) and 3)–iii), the open set $V_R-\{f=0\}$ is a $G_R^+$-orbit and we can define $|f|^s(x)$ in the same way as (9) for a generic $s\in C$. The Fourier transform of $|f|^s(x)$ is as follows:

$$(12)\quad \int |f|^s(x)\exp(-\sqrt{-1}\langle x, x'\rangle)dx$$
$$=(2\pi)^{3l+2}\cdot\Gamma(s+1)\Gamma\left(s+\frac{l+3}{2}\right)\Gamma\left(s+\frac{2l+3}{2}\right)\Gamma\left(s+\frac{3l+4}{2}\right)4^{2s+n/4}$$
$$\cdot 4\cdot\sin(\pi s)\cdot\cos(\pi s)\cdot|f|^{-s-(n/4)}(x'),$$

for $l=2$ and $4$ in the case of 1)–iii) and 3)–iii), respectively.

## References


[1] J. Igusa: A classification of spinors up to dimension twelve. Amer. J. Math., **92**, 997–1028 (1970).

[2] N. Jacobson: Exceptional Lie Algebras. Dekker (1971) (lecture note).

[3] T. Kimura: The $b$-functions and holonomy diagrams of irreducible prehomogeneous vector spaces (preprint).

[4] M. Sato and T. Kimura: A classification of irreducible prehomogeneous vector spaces and their relative invariants. Nagoya. Math. J., **65**, 1–155 (1977).

[5] M. Sato and T. Shintani: On zeta functions associated with prehomogeneous vector spaces. Ann. of Math., **100**, 131–170 (1974).