12. *Quasi-Periodic Solutions of Nonlinear Equations of sine-Gordon Type and Fixed Point Free Involutions of Hyperelliptic Curves*

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1. The connection between the theory of nonlinear equations solvable by the inverse scattering method and the theory of Riemann surfaces (algebraic curves) has been discussed by many authors (see, for example, [6], [3], [8], [9]).

In this note we consider quasi-periodic solutions of the sine-Gordon equation

(1) $$u_{\xi\eta}+\sin u=0$$

and the equation of the system of Pohlmeyer [10] and Lund-Regge [7]

(2) $$\begin{aligned}&u_{\xi\eta}-v_\xi v_\eta \sin(u/2)/2\cos^3(u/2)+\sin u=0,\\ &v_{\xi\eta}+(u_\xi v_\eta+u_\eta v_\xi)/\sin u=0\end{aligned}$$

which is a generalization of (1); if v=constant, then (2) reduces to (1). We show that the solutions of (2) which correspond to the hyperelliptic curves admitting fixed point free involutions reduce to the solutions of (1).

Quasi-periodic solutions of (1) were discussed by Kozel-Kotlyarov [5] and by Its [4]. With the aid of the representation of (1) as the compatibility condition of the linear differential equations

(3) $$\begin{aligned}&i\Psi_\xi+2^{-1}u_\xi\begin{pmatrix}1&0\\0&-1\end{pmatrix}\Psi+2^{-1}\zeta\begin{pmatrix}0&1\\1&0\end{pmatrix}\Psi=0,\\ &i\Psi_\eta+2^{-1}\zeta^{-1}\begin{pmatrix}0&\exp(iu)\\ \exp(-iu)&0\end{pmatrix}\Psi=0,\qquad \zeta\in C,\end{aligned}$$

they showed that quasi-periodic solutions of (1) correspond to the hyperelliptic curves

(4) $$w^2=z\prod_{j=1}^{2g}(z-z_j).$$

They also showed that the simultaneous solution Ψ of (3) and the parameter ζ in (3) are two-valued functions on the Riemann surface of (4).

First we construct quasi-periodic solutions of (2) by a method similar to that of Kricheber [6], starting from the curves $\mu^2=\prod_{j=1}^{2g+2}(\lambda-\lambda_j)$. Next we show that the solutions of (2) constructed in this way reduce to the solutions of (1) when we specialize the curves to the form

(5) $$\mu^2=\prod_{j=1}^{2g}(\lambda-\lambda_j)(\lambda+\lambda_j).$$

This assertion is proved by using a fixed point free involution (λ,μ)

$\mapsto(-\lambda, -\mu)$ of the curves (5). In particular, our treatment explains naturally the two-valuedness of Ψ and ζ in (3).

We note that our solutions are complex-valued in general. The reality conditions will be discussed elsewhere together with the detailed proofs of the results of this note. The classical massive Thirring model which we discussed in [2] can be also treated by the method in this note.

After completion of the present work, a paper of Cherednik's [1] was published, in which quasi-periodic solutions of a class of equations including (2) are treated, but the reductions used in this note are not discussed.

2. Construction of solutions of (2). The equation (2) is the compatibility conditions of the linear differential equations

$$
\begin{aligned}
&i\Phi_\xi+\begin{pmatrix}0 & a^*\\ a & 0\end{pmatrix}\Phi+2^{-1}\lambda\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Phi=0,\\
&i\Phi_\eta+2^{-1}\lambda^{-1}\begin{pmatrix}\cos u & -\sin u\exp(-i\omega)\\ -\sin u\exp(i\omega) & -\cos u\end{pmatrix}\Phi=0
\end{aligned}\tag{6}
$$

where $a=i(\sin u\exp(i\omega))_\xi/2\cos u$, $\omega_\xi=v_\xi\cos u/2\cos^2(u/2)$, $\omega_\eta=v_\eta/2\cos^2(u/2)$, λ is a complex parameter and a^* is the complex conjugate of a ([10]).

Let $\boldsymbol{S}$ be the Riemann surface of the hyperelliptic curve $\mu^2=\prod_{j=1}^{2g+2}(\lambda-\lambda_j)$, $\lambda_j\neq\lambda_k$ $(j\neq k)$, $\lambda_j\neq 0$ of genus $g>0$. Denote by $\boldsymbol{P}_1, \boldsymbol{P}_2$ (resp. $\boldsymbol{Q}_1, \boldsymbol{Q}_2$) the points on $\boldsymbol{S}$ whose projections on $\boldsymbol{CP}^1$ by λ are ∞ (resp. 0). As local parameters around $\boldsymbol{P}_j$ (resp. $\boldsymbol{Q}_j$) we take λ^{-1} (resp. λ). Let δ be a positive divisor of degree $g+1$ on $\boldsymbol{S}$ such that $\ell(\delta-\boldsymbol{P}_j)=1$.

By using the theory of abelian integrals on $\boldsymbol{S}$, we can construct functions $\Phi_j(\xi,\eta,\boldsymbol{P})$, $j=1,2$, $(\xi,\eta)\in\boldsymbol{R}^2$, $\boldsymbol{P}\in\boldsymbol{S}$ with the following properties:

i) Φ_j are meromorphic on $\boldsymbol{S}-\{\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{Q}_1,\boldsymbol{Q}_2\}$ and whose pole divisors are δ,

ii) around $\boldsymbol{P}_k$ (resp. $\boldsymbol{Q}_k$) Φ_j are expanded as

$$
\begin{aligned}
\Phi_j(\xi,\eta,\boldsymbol{P})=&(\text{Taylor series in }\lambda^{-1}\text{ with constant terms }\delta_{jk})\\
&\times\exp(2^{-1}a_k\xi\lambda),\qquad a_1=1,\ a_2=-1
\end{aligned}\tag{7}
$$

$$
(\text{resp. }\Phi_j(\xi,\eta,\boldsymbol{P})=(\text{Taylor series in }\lambda)\exp(2^{-1}\lambda^{-1}a_k\eta)).\tag{8}
$$

Further the function $\Phi={}^t(\Phi_1,\Phi_2)$ satisfies the equations

$$
\begin{aligned}
&i\Phi_\xi+\begin{pmatrix}0 & -\alpha_{12}\\ \alpha_{21} & 0\end{pmatrix}\Phi+2^{-1}\lambda\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Phi=0,\\
&i\Phi_\eta+\frac{1}{2\lambda(\beta_{11}\beta_{22}-\beta_{12}\beta_{21})}\begin{pmatrix}\beta_{11}\beta_{22}+\beta_{12}\beta_{21} & -2\beta_{11}\beta_{12}\\ 2\beta_{21}\beta_{22} & -\beta_{11}\beta_{22}-\beta_{12}\beta_{21}\end{pmatrix}\Phi=0
\end{aligned}
$$

where α_{jk} (resp. β_{jk}) are the coefficients of λ^{-1} (resp. λ^0) in (7) (resp. (8)).

Comparing these equations with the equations (6), we conclude that the pair of functions

$$(9)\qquad \begin{aligned} &u=\arccos\left((\beta_{11}\beta_{22}+\beta_{12}\beta_{21})/(\beta_{11}\beta_{22}-\beta_{12}\beta_{21})\right),\\ &v=i\log(\beta_{12}/\beta_{21})+v_0, \qquad v_0\in \boldsymbol{C}\end{aligned}$$

is a solution of (2).

These solutions are expressed by theta functions on $\boldsymbol{S}$ (cf. [6]).

3. Involutions and solutions of (1). We specialize the curves used in § 2 to the form $\mu^2=\prod_{j=1}^{2g}(\lambda-\lambda_j)(\lambda+\lambda_j)$. This curve admits a fixed point free involution $\boldsymbol{T}:(\lambda,\mu)\mapsto(-\lambda,-\mu)$. By using the effect of $\boldsymbol{T}$ on abelian integrals and theta functions on $\boldsymbol{S}$ (see Rauch-Farkus [11, Chap. 6]) and by taking the divisor δ such that $\boldsymbol{T}\delta=\delta$, we can show that the relations

$$\Phi_1(\xi,\eta,\boldsymbol{TP})=\Phi_2(\xi,\eta,\boldsymbol{P}),\quad \alpha_{12}=-\alpha_{21},\quad \beta_{11}=\beta_{22},\quad \beta_{12}=\beta_{21}$$

hold. In view of (9), we have $v=$constant. That is, we obtain a solution of (1).

In order to recover the linear equations (3), we put

$$\Psi={}^t(\Psi_1,\Psi_2),\quad \Psi_1=\Phi_1+\Phi_2,\quad \Psi_2=\Phi_1-\Phi_2.$$

Then the function Ψ satisfies the equations

$$i\Psi_\xi+\begin{pmatrix}\alpha_{21} & 0\\ 0 & -\alpha_{21}\end{pmatrix}\Psi+2^{-1}\lambda\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Psi=0,$$

$$i\Psi_\eta+2^{-1}\lambda^{-1}\begin{pmatrix}0 & (\beta_{11}+\beta_{12})/(\beta_{11}-\beta_{12})\\ (\beta_{11}-\beta_{12})/(\beta_{11}+\beta_{12}) & 0\end{pmatrix}\Psi=0.$$

Remark. The function Ψ_1 (resp. Ψ_2) is invariant (resp. anti-invariant) under $\boldsymbol{T}$, therefore Ψ_1 (resp. Ψ_2) is single-valued (resp. two-valued) on the curve $w^2=z\prod_{j=1}^{2g}(z-\lambda_j^2)$, $w=\lambda\mu$, $z=\lambda^2$, which is the quotient of the curve $\mu^2=\prod_{j=1}^{2g}(\lambda-\lambda_j)(\lambda+\lambda_j)$ by $\boldsymbol{T}$. These facts and the fact that λ is two-valued on the curve $w^2=z\prod_{j=1}^{2g}(z-\lambda_j^2)$ recover the construction of quasi-periodic solutions of (1) by Its [4].

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