106. *On Lévy's Downcrossing Theorem*

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Let $\{X_t, t\geqq 0\}$ be a one-dimensional standard Brownian motion starting at 0. For $\varepsilon>0$ and $t>0$, put $\sigma_0(=\sigma_0^\varepsilon)=0$, $\tau_n(=\tau_n^\varepsilon)=\inf\{s>\sigma_n; |X_s|=\varepsilon\}$, $\sigma_{n+1}(=\sigma_{n+1}^\varepsilon)=\inf\{s>\tau_n; |X_s|=0\}$ $(n=0, 1, 2, \cdots)$ and $d_\varepsilon(t)=\max\{n; \sigma_n^\varepsilon\leqq t\}$. Thus $d_\varepsilon(t)$ is the number of times that the reflected Brownian motion $|X_s|$ crosses down from ε to 0 by time t.

Lévy's downcrossing theorem.

$$P\left(\lim_{\varepsilon\downarrow 0}\varepsilon d_\varepsilon(t)=l(t),\ t\geqq 0\right)=1 \tag{1}$$

where $l(t)$ is the local time of $X(t)$ at 0, i.e.,

$$l(t)=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(X_s)ds \qquad a.s. \tag{2}$$

(For the original version of this theorem, see p. 48 of [4] or [1], [7].)

The aim of this article is to prove a *central limit theorem*;

Theorem. *D-valued processes $1/\sqrt{\varepsilon}\cdot(\varepsilon d_\varepsilon(t)-l(t))$ converge in law to $B(l(t))$ as $\varepsilon\downarrow 0$, where $B(\cdot)$ is a Brownian motion independent of $l(\cdot)$ $(B_0=0)$.*

This is a natural assertion because $d_\varepsilon(l^{-1}(t))$ is Poisson distributed with mean $t\varepsilon^{-1}$ (see D. Williams [7]). However, the difficulty is that the independence of B and l does not seem to be trivial from this kind of argument. Therefore we will use another approach based on the following lemma due to D. Stroock*).

Lemma 1. *Let*

$$\theta_t^\varepsilon=\sum_n 1_{[\sigma_n,\tau_n)}(t)\,\mathrm{sgn}\,(X_t)1_{(-\varepsilon,\varepsilon)}(X_t),$$

then,

$$\left|\varepsilon d_\varepsilon(t)-l(t)-\int_0^t\theta_s^\varepsilon dX_s\right|\leqq\varepsilon \qquad a.s. \tag{3}$$

Since the proof is not published yet, we will prove it for the convenience of the reader. By the generalized Itô's formula (Tanaka's formula),

$$\begin{aligned}\varepsilon&=|X_{\tau_n}|-|X_{\sigma_n}|\\&=\int_{\sigma_n}^{\tau_n}\mathrm{sgn}\,(X_t)dX_t+l(\tau_n)-l(\sigma_n)\\&=\int_{\sigma_n}^{\sigma_{n+1}}\theta_t^\varepsilon dX_t+l(\sigma_{n+1})-l(\sigma_n).\end{aligned}$$

*) Private communication.

Combining this with

$$l(t)+\int_0^t \theta_s^\varepsilon dX_s=\sum\left\{\int_{\sigma_n\wedge t}^{\sigma_{n+1}\wedge t}\theta_s^\varepsilon dX_s+l(\sigma_{n+1}\wedge t)-l(\sigma_n\wedge t)\right\}$$

we have the assertion.

Remark. We can prove (1) using the previous lemma. Since $|\theta_t^\varepsilon|\leq 1_{(-\varepsilon,\varepsilon)}(X_t)$, it is easy to see

$$(4)\qquad \lim_{n\to\infty}\int_0^t \theta_s^{\varepsilon_n} dX_s=0 \quad \text{a.s.}\quad \text{where } \varepsilon_n=n^{-2}.$$

Therefore, $\lim_{n\to\infty}\varepsilon_n d_{\varepsilon_n}(t)=l(t)$ a.s. Since $d_\varepsilon(t)$ is monotonic in ε, it is not difficult to complete the proof of (1).

We return to the proof of Theorem. Thanks to Lemma 1, it suffices to show that the continuous martingales

$$M_t^\varepsilon=\frac{1}{\sqrt{\varepsilon}}\int_0^t \theta_s^\varepsilon dX_s$$

converge in law to $B(l(t))$ as $\varepsilon\to 0$.

Lemma 2. *There exist constants c_n and C_n, $n=1,2,\cdots$ such that*

$$E[(M_t^\varepsilon-M_s^\varepsilon)^{2n}]\leq c_n E[(\langle M^\varepsilon\rangle(t)-\langle M^\varepsilon\rangle(s))^n]$$
$$\leq C_n(t-s)^n,\qquad t\geq s\geq 0.$$

Proof. The first inequality is well known (see [3]). To see the second inequality, notice that

$$E[(\langle M^\varepsilon\rangle(t)-\langle M^\varepsilon\rangle(s))^n]=E\left[\left(\frac{1}{\varepsilon}\int_s^t(\theta_u^\varepsilon)^2du\right)^n\right]$$
$$\leq E\left[\left(\frac{1}{\varepsilon}\int_s^t 1_{(-\varepsilon,\varepsilon)}(X_u)du\right)^n\right].$$

Since X_t is a Brownian motion, we easily have the assertion by a direct computation.

Lemma 3. *For all $n=1,2,\cdots$, and $t>0$,*

$$(5)\qquad \lim_{\varepsilon\downarrow 0}E[(\langle M^\varepsilon\rangle(t)-l(t))^{2n}]=0$$

$$(6)\qquad \lim_{\varepsilon\downarrow 0}E[(\langle M^\varepsilon, X\rangle(t))^{2n}]=0.$$

Proof. By Itô's formula,

$$\begin{aligned}\varepsilon^2&=X_{\tau_n}^2-X_{\sigma_n}^2\\&=2\int_{\sigma_n}^{\tau_n}X_u dX_u+\int_{\sigma_n}^{\tau_n}du\\&=2\int_{\sigma_n}^{\sigma_{n+1}}|X_u|\theta_u^\varepsilon dX_u+\int_{\sigma_n}^{\sigma_{n+1}}|\theta_u^\varepsilon|^2du.\end{aligned}$$

Therefore,

$$(7)\qquad \left|\varepsilon^2 d_\varepsilon(t)-2\int_0^t|X_u|\theta_u^\varepsilon dX_u-\int_0^t(\theta^\varepsilon)^2du\right|\leq\varepsilon^2.$$

Combining (7) and (3), we have

$$(8)\qquad \left|l(t)+\int_0^t\theta_u^\varepsilon dX_u-\frac{2}{\varepsilon}\int_0^t|X_u|\theta_u^\varepsilon dX_u-\frac{1}{\varepsilon}\int_0^t(\theta_u^\varepsilon)^2du\right|\leq 2\varepsilon.$$

Here notice that

$$|\theta_u^\varepsilon| \leqq 1_{(-\varepsilon,\varepsilon)}(X_u) \quad \text{and} \quad \frac{1}{\varepsilon}|X_u\theta_u^\varepsilon| \leqq 1_{(-\varepsilon,\varepsilon)}(X_u).$$

Therefore, the $2n^{th}$ moments of the second and the third terms of (8) are of order ε^n (see Lemma 2). Thus we have (5). (6) can be proved in a similar (but easier) way.

We are now ready to prove the theorem. Let P^ε denote the probability measure on $\Omega = C([0,\infty)\to R^3)$ induced by $(X_t, M_t^\varepsilon, l(t))$. We will use $(x(t), y(t), z(t))$ to express elements of Ω. By Lemma 2, $\{P^\varepsilon : \varepsilon > 0\}$ is precompact. Let P^* be any limit point of $\{P^\varepsilon : \varepsilon > 0\}$. Then it is clear that $x(t)$ and $y(t)$ are martingales relative to $(P^*, \mathcal{F}_t; t > 0)$ where $\{\mathcal{F}_t\}$ is the natural increasing family of σ-fields. Of course, clearly, $x(t)$ is an $\mathcal{F}_t$-Brownian motion with local time $z(t)$. By Lemma 3, we also see that $\langle x\rangle(t) = z(t)$, $\langle x, y\rangle(t) = 0$, P^*-a.s. Therefore, by the Knight representation theorem for continuous martingales (see [5]) there exists a two-dimensional Brownian motion $(B_1(t), B_2(t))$ such that $x(t) = B_1(t)$ and $y(t) = B_2(\langle y\rangle_t) = B_2(z(t))$, a.s. Since P^* is unique, we have the theorem by a standard argument.

Remark. (a) In a similar way, we can prove the following for the number of downcrossings of X_t (instead of $|X_t|$) (cf. [2]). For $a < 0 < b$, let $D_{a,b}(t)$ denote the number of times that X_s crosses down from b to a by time t. Then,

(1)′ $$\lim_{n\to\infty} 2(b_n - a_n) D_{a_n,b_n}(t) = l(t) \quad \text{a.s.,} \quad \text{if} \quad \sum_n (b_n - a_n) < \infty.$$

(9) $$\sqrt{(b_n - a_n)/2(b_n^2 + a_n^2)}\,\{2(b_n - a_n) D_{a_n,b_n}(t) - l(t)\}$$

converge in law to $B(l(t))$ as $a_n, b_n \to 0$, where $l(t)$ and $B(t)$ are the same as in Theorem.

(b) As a corollary of Papanicolaou-Stroock-Varadhan's theorem ([6]), we can also prove a central limit theorem for (2):

(10) $$\frac{1}{\sqrt{\varepsilon}}\left\{\frac{1}{2\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(X_s)ds - l(t)\right\}$$

converge in law to

$$\sqrt{2/3}\,B(l(t)), \qquad \text{as } \varepsilon \to 0.$$

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