76. *Some Results in the Classification Theory of Compact Complex Manifolds in* $\mathcal{C}$

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0. By definition a compact complex manifold X is in $\mathcal{C}$ if there exist a compact Kähler manifold Z and a surjective meromorphic map $h: Z \to X$ [1]. The purpose of this note is then to report some results on the structure of manifolds in $\mathcal{C}$. Details will appear elsewhere.

In what follows X, Y, Z, etc. always denote compact connected complex manifolds in $\mathcal{C}$. We set $q(X) = \dim H^1(X, O_X)$ and $a(X) =$ the algebraic dimension of X [2]. Let $f: X \to Y$ be a holomorphic map. For any open subset $U \subseteqq Y$ we write $X_U = f^{-1}(U)$ and $f_U = f|_{X_U}$, and write $X_y = f^{-1}(y)$ for $y \in Y$. We call f a *fiber space* if f is proper and surjective and has connected fibers. Suppose that f is a fiber space. Then we set $\dim f = \dim X - \dim Y$, and $q(f) = q(X_y)$ for any smooth fiber X_y. Further any fiber space $f^*: X^* \to Y^*$ which is bimeromorphic to f is called a *bimeromorphic model* of f.

1. Let $f: X \to Y$ be a fiber space and U a Zariski open subset of Y over which f is smooth. For any integer $k \geqq 0$ we set $A_k = \{u \in U; a(X_u) \geqq k\}$.

Proposition 1. *A_k is a union of at most countably many analytic subsets of U whose closures are analytic in Y.*

Let $a(f) = \max\{k; A_k = U\}$, so that $a(X_u) = a(f)$ for 'general' $u \in U$. We call $a(f)$ the *relative algebraic dimension* of f. By Proposition 1 $a(f)$ depends only on the bimeromorphic equivalence class of f. Clearly $0 \leqq a(f) \leqq \dim f$.

Proposition 2. *Let $f: X \to Y$ and U be as above. Then the following conditions are equivalent.* 1) $a(f) = \dim f$, 2) $f_U: X_U \to U$ *is locally Moishezon, and* 3) *there exists a bimeromorphic model $f^*: X^* \to Y^*$ of f which is locally Moishezon.*

Here a morphism $g: X \to Y$ is called *locally Moishezon* if for each $y \in Y$ there exists a neighborhood $y \in V$ such that $g_V: X_V \to V$ is Moishezon, i.e., bimeromorphic over V to a projective morphism.

Definition 1. Let $f: X \to Y$ be a fiber space. Then a *relative algebraic reduction* of f is a commutative diagram

$$\begin{tikzcd} X^* \arrow[rr, "f^*"] \arrow[dr, "g"'] & & Y^* \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

where f^* is a bimeromorphic model of f and $a(f)=a(h)=\dim h$. We also call g a relative algebraic reduction of f. When Y is a point, this reduces to the usual definition of an algebraic reduction of X [2].

Theorem 1. *Let $f: X \to Y$ be any fiber space. Then there exists a relative algebraic reduction of f, and up to bimeromorphic equivalence it is unique.*

Applying Theorem 1 successively, for any fiber space f we can find a bimeromorphic model $f^*: X^* \to Y^*$ which admits a decomposition;

$$(1)\qquad \begin{tikzcd} X^* \arrow[rrrr, dotted, "f^*"] \arrow[d, equal] & & & & Y^* \arrow[d, equal] \\ X^* \arrow[r, "g"'] & Y_m \arrow[r, "h_m"'] & \cdots \arrow[r, "h_2"'] & Y_1 \arrow[r, "h_1"'] & Y_0 \end{tikzcd}$$

where 1) $h_i: Y_i \to Y_{i-1}$, $1 \leqq i \leqq m$, are locally Moishezon, 2) $a(g)=0$, and 3) the following commutative diagram is an algebraic reduction of $h^i = h_i \cdots h_m g$

$$\begin{tikzcd} X \arrow[rr, "h^i"] \arrow[dr, "h^{i+1}"'] & & Y_{i-1} \\ & Y_i \arrow[ur, "h_i"'] & \end{tikzcd}.$$

(The case where $m=0$, or g is isomorphic is included.) Moreover f^* and the diagram (1) are up to bimeromorphic equivalence uniquely determined by f.

We shall call the diagram (1) the *canonical decomposition* of f.

2. Definition 2. Let $f: X \to Y$ be a fiber space. Then a *relative Albanese map* for f is a commutative diagram

$$(2)\qquad \begin{tikzcd} X^* \arrow[rr, "f^*"] \arrow[dr, "\alpha^*"'] & & Y^* \\ & \mathrm{Alb}\, X^*/Y^* \arrow[ur] & \end{tikzcd}$$

with the following properties; 1) f^* is a bimeromorphic model of f and 2) there exists a Zariski open subset $U \subseteqq Y^*$ such that f^* is smooth over U and that for each $u \in U$ the induced map $\alpha_u: X_u^* \to \mathrm{Alb}\,(X^*/Y^*)_u$ is bimeromorphic to the Albanese map $\alpha(u): X_u^* \to \mathrm{Alb}\, X_u^*$.

Consider a commutative diagram of fiber spaces

$$\begin{tikzcd} X \arrow[rrr, "g"] \arrow[dr, "f"'] \arrow[drr, "f'"] & & & Z \\ & Y \arrow[r] & Y' \arrow[ur] & \end{tikzcd}$$

where f' is a relative algebraic reduction of g.

Theorem 2. *In the above situation there exists a relative Albanese map* (2) *for f for which α^* is a fiber space. Moreover it is unique up to bimeromorphic equivalence.*

Corollary. *Let $f: X \to Y$ be a fiber space. Then there exists a relative Albanese map* (2) *for f with α^* a fiber space if either f is an algebraic reduction of X, or $a(X)=0$.*

By Theorem 2 h_i in the canonical decomposition (1) admits a relative Albanese map

$$\begin{tikzcd} Y_i^* \arrow[rr, "h_i^*"] \arrow[dr, "\alpha_i"'] & & Y_{i-1}^* \\ & \mathrm{Alb}\,(Y_i^*/Y_{i-1}^*) \arrow[ur, "\mu_i"'] & \end{tikzcd} \quad \begin{array}{l} 1 \leqq i \leqq m+1 \\ (h_{m+1}=g) \end{array} \tag{3}$$

where α_i is a fiber space. (For h_1 we have to assume that f^* is an algebraic reduction of X.) The following two theorems give informations on the structure of μ_1 and α_i, while they are of independent interest.

Theorem 3. *Let $f: X \to Y$ be a fiber space. Suppose that f is locally Moishezon and $q(f)=0$. Then f is Moishezon, i.e., bimeromorphic over Y to a projective morphism.*

Let $f: X \to Y$ be an algebraic reduction of X. Then by Corollary above there exists a relative Albanese map (2) for f with α^* a fiber space.

Theorem 4. *Suppose that f is locally Moishezon. Then $q(\alpha^*)=0$.*

Corollary. *In* (3) *we have* 1) $\dim \mu_i > 0$, 2) $q(\alpha_i)=0$, *and* 3) *α_i is Moishezon.*

We further note that we can take (3) in such a way that α_{iu} is isomorphic to $\alpha_i(u)$ in the notation of Definition 2 for $i \leqq m$ (due to the local Moishezonness of h_i).

3. Let $ca(X) = \dim X - a(X)$. Clearly $ca(X)=0$ if and only if X is Moishezon. When $ca(X)=1$, the general fiber of the algebraic reduction of X is an elliptic curve, as is well-known.

Theorem 5. *Suppose that $ca(X)=2$. Then there exists an algebraic reduction $f^*: X^* \to Y^*$ of X such that if we denote by X_y^* any smooth fiber of f^*, then one of the following holds:* 1) *X_y^* is a complex torus,* 2) *X_y^* is a K3 surface and $a(X_y^*)=0$ for general $y \in Y$, or* 3) *X_y^* is an almost homogeneous ruled surface of genus* 1 *which is relatively minimal* (cf. Remark 12.5 of [2]).

4. We consider manifolds of algebraic dimension zero. So assume that $a(X)=0$. In this case the Albanese map $\alpha: X \to \mathrm{Alb}\,X$ is a fiber space [2]. In particular $q(X) \leqq \dim X$. We set $q^*(X) = \max\{q(\tilde{X}); \tilde{X}$ a finite unramified covering of $X\}$ and $q^{**}(X) = \max\{q(\tilde{X}); \tilde{X}$ a finite (ramified) covering of $X\}$. By the above remark we get $q(X) \leqq q^*(X) \leqq q^{**}(X) \leqq \dim X$. We call X *primary* (*specially primary*) if $q(X)=q^*(X)$ ($q(X)=q^{**}(X)$). Clearly each X has a finite unramified covering X which is primary. Now we consider a relative algebraic reduction (4) of α:

$$
(4) \qquad \begin{tikzcd} X^* \arrow[rr, "\alpha^*"] \arrow[dr, "g"'] & & \mathrm{Alb}\, X \\ & Z \arrow[ur, "h"'] & \end{tikzcd} .
$$

Theorem 6. *Suppose that X is primary. Then the following holds true.* 1) $q(\alpha^*)(=q(\alpha))=0$, 2) *there exists a Zariski open subset U of Y such that $h_U: Z_U \to U$ is a holomorphic fiber bundle whose typical fiber is a unirational, almost homogeneous projective manifold, and* 3) $a(g)=0$, *so that* (4) *gives the canonical decomposition* (1) *of α. Moreover if X is specially primary, then $q(g)=0$.*

Let $cq(X)=\dim X - q(X)$. Suppose that X is primary. Then $cq(X)>0$. If $cq(X)=1$, then α is a holomorphic $\boldsymbol{P}^1$ bundle over some Zariski open subset U of $\mathrm{Alb}\, X$, as follows from the above theorem. In general let $\kappa(X)$ be the Kodaira dimension of X [2]. Since $a(X)=0$, $\kappa(X)=0$ or $-\infty$.

Theorem 7. *Suppose that $cq(X)=2$.* 1) *If $\kappa(X)=0$, then there exists a finite unramified covering $\tilde{X}$ of X which is bimeromorphic to a product* $\mathrm{Alb}\, \tilde{X} \times S$ *where S is a $K3$ surface with $a(S)=0$.* 2) *If $\kappa(X)=-\infty$, then there exists a Zariski open subset $U \subseteqq \mathrm{Alb}\, X$ such that $\alpha_U: X_U \to U$ is a holomorphic fiber bundle whose typical fiber is an almost homogeneous rational surface.*

References

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