

56. Analytic Expressions of Unstable Manifolds

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§ 0. Introduction. In the study of bifurcations of differentiable dynamical systems, topological configurations of stable manifolds and unstable manifolds play an important role. In this note we give global analytic expressions by analytic mappings for unstable sets of strictly hyperbolic fixed points of analytic mappings $f: R^n \rightarrow R^n$. If mapping f is a diffeomorphism, the obtained unstable set agrees with the unstable manifold of the fixed point.

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§ 1. Main theorems. Let $f: R^n \rightarrow R^n$ be a real analytic map defined globally on R^n . We assume that the origin, O , is a fixed point of f , i.e., $f(O) = O$, and that the Jacobian matrix df_o at O is diagonalisable.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ denote the eigenvalues of df_o . We assume O is hyperbolic, i.e.,

$$(1) \quad \begin{cases} |\alpha_i| > 1 & \text{for } i=1, 2, \dots, k, \\ |\alpha_i| < 1 & \text{for } i=k+1, \dots, n. \end{cases}$$

Let $\delta = (\delta_1, \dots, \delta_k)$ be multi-index with $\delta_i \geq 0$ for $i=1, \dots, k$. Let $\alpha = (\alpha_1, \dots, \alpha_k)$. We denote $|\delta| = \delta_1 + \delta_2 + \dots + \delta_k$ and $\alpha^\delta = \alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdot \dots \cdot \alpha_k^{\delta_k}$. We assume also

$$(2) \quad \alpha^\delta \neq \alpha_i$$

for any multi-index δ with $|\delta| \geq 2$ and $i=1, \dots, k$.

Let E^u denote the subspace of tangent space $T_o R^n$ spanned by the eigenvectors for eigenvalues $\alpha_1, \dots, \alpha_k$. Space E^u is invariant under the differential map $df_o: T_o R^n \rightarrow T_o R^n$. Let $\eta: E^u \rightarrow E^u$ be the differential map df_o restricted on E^u , i.e.,

$$\eta(\xi) = df_o(\xi) \quad \text{for } \xi \in E^u.$$

We call a point P in R^n an unstable point of o if there is a sequence of points $P_i \in R^n$, $i=0, -1, -2, \dots$, such that $P_i = f(P_{i-1})$ for $i=0, -1, -2, \dots$, $P = P_0$ and that P_i tends to the origin as i tends to $-\infty$. We denote the set of unstable points of O by W^u . We call W^u the unstable set of O . If f is a diffeomorphism, then W^u is nothing but the unstable manifold of O .

Theorem 1. *Let $f: R^n \rightarrow R^n$ be a real analytic map defined globally*

on R^n , with $f(O)=O$. Assume that the Jacobian matrix df_o at O is diagonalisable and that the eigenvalues $\alpha_1, \dots, \alpha_n$ satisfy conditions (1) and (2). Then there is a real analytic map $\phi: E^u \rightarrow R^n$ defined globally on E^u satisfying the following conditions:

- i) $\phi(O)=O$,
- ii) $d\phi_0: T_oE^u = E^u \rightarrow T_oR^n$ is the inclusion map,
- iii) $\phi(E^u) = W^u$,
- iv) $f \circ \phi = \phi \circ \eta$,
- v) Taylor coefficients of ϕ are given by Theorem 2 below.

In order to give the formula for Taylor coefficients, we introduce several notations. As we have assumed that df_o is diagonalisable with eigenvalues $\alpha_1, \dots, \alpha_n$, we can find a system of coordinates $x=(x_1, \dots, x_n)$ of R^n such that

$$df_o = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix}.$$

Let $d=(d_1, \dots, d_n)$ be a multi-index with $d_i \geq 0$ for $i=1, \dots, n$. Let $|d|=d_1 + \dots + d_n$. For $x=(x_1, \dots, x_n)$ we denote

$$x^d = x_1^{d_1} \cdot x_2^{d_2} \cdot \dots \cdot x_n^{d_n}.$$

Let $f(x)=(f_1(x), f_2(x), \dots, f_n(x))$ and

$$f_i(x) = \alpha_i x_i + \sum_{|d| \geq 2} f_{i,d} x^d \quad \text{for } i=1, \dots, n.$$

As for multi-indexes of length 1, we denote

$$\delta(i) = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$$

and

$$d(i) = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0).$$

Let $f_{i,d(i)} = \alpha_i$ for $i=1, \dots, n$ and $f_{i,d(j)} = 0$ for $i \neq j$. We have $df_o = (f_{i,d(j)})$. The space E^u is spanned by vectors $\partial/\partial x_i$, $i=1, \dots, k$. Denote by $\xi=(\xi_1, \dots, \xi_k)$ the coordinate on E^u with basis $\partial/\partial x_i$, $i=1, \dots, k$. We identify linear spaces E^u and R^k by this coordinate system. For $\xi=(\xi_1, \dots, \xi_k)$ and multi-index $\delta=(\delta_1, \dots, \delta_k)$ we denote

$$\xi^\delta = \xi_1^{\delta_1} \cdot \xi_2^{\delta_2} \cdot \dots \cdot \xi_k^{\delta_k}.$$

Let $\phi: E^u \rightarrow R^n$ be a real analytic map with $\phi(O)=O$. Let $\phi(\xi)=(\phi_1(\xi), \phi_2(\xi), \dots, \phi_n(\xi))$ and

$$\phi_i(\xi) = \sum_{|\delta| \geq 1} \phi_{i,\delta} \xi^\delta.$$

Let $\phi_{i,\delta(i)} = 1$ for $i=1, \dots, k$ and let $\phi_{i,\delta(j)} = 0$ for $i=1, \dots, n$ and $j=1, \dots, k$ with $i \neq j$. Let $\lambda(p, q)$ denote the set of multi-indexes $\delta=(\delta_1, \dots, \delta_k)$ with $p \leq |\delta| \leq q$. For ϕ_i , a positive integer p and a multi-index δ , we put

$$\theta(\phi_i, p, \delta) = \sum_{\substack{\gamma_1, \dots, \gamma_p \in \lambda(1, |\delta|) \\ \gamma_1 + \dots + \gamma_p = \delta}} \phi_{i,\gamma_1} \cdot \phi_{i,\gamma_2} \cdot \dots \cdot \phi_{i,\gamma_p}.$$

Note that if $p \geq 2$, $\theta(\phi_i, p, \delta)$ contains no $\phi_{i,\gamma}$ satisfying $|\gamma| \geq |\delta|$. We have

$$(\phi_i(\xi))^p = \sum_{\delta \in \lambda(p, \infty)} \xi^\delta \theta(\phi_i, p, \delta).$$

For multi-indexes $\delta = (\delta_1, \dots, \delta_k)$ and $d = (d_1, \dots, d_n)$, let

$$\Gamma(\phi, \delta, d) = \sum_{\substack{r_1, \dots, r_n \in \lambda(1, |\delta|) \\ r_1 + \dots + r_n = \delta}} \left(\prod_{i=1}^n \theta(\phi_i, d_i, r_i) \right).$$

Note that $\Gamma(\phi, \delta, d)$ contains no $\phi_{i,r}$ with $|r| \geq |\delta|$ if $|d| \geq 2$. We have

$$(\phi(\xi))^d = \sum_{\delta \in \lambda(|d|, \infty)} \xi^\delta \Gamma(\phi, \delta, d).$$

Let $l(p, q)$ denote the set of multi-indexes $d = (d_1, \dots, d_n)$ satisfying $p \leq |d| \leq q$. Using the notations defined above, we obtain the expressions

$$f_i(\phi(\xi)) = \sum_{\delta \in \lambda(1, \infty)} \xi^\delta \left(\sum_{d \in l(1, |\delta|)} f_{i,d} \Gamma(\phi, \delta, d) \right).$$

Theorem 2. *The Taylor coefficients $\phi_{i,\delta}$ of mapping ϕ in Theorem 1 are computed as follows:*

- i) for multi-index δ with $|\delta| = 1$,
 $\phi_{i,\delta(i)} = 1$ for $i = 1, \dots, k$,
 $\phi_{i,\delta(j)} = 0$ for $i = 1, \dots, n, j = 1, \dots, k$, with $i \neq j$,
- ii) for multi-index δ with $|\delta| \geq 2$, define $\phi_{i,\delta}$ inductively by the formula:

$$\phi_{i,\delta} = \frac{1}{\alpha^\delta - \alpha_i} \left(\sum_{d \in l(2, |\delta|)} f_{i,d} \Gamma(\phi, \delta, d) \right).$$

The mapping ϕ can be extended to an analytic map on E^u .

§ 2. Sketch of the proof. If conditions (2) is satisfied, starting from i) in Theorem 2, we can compute $\phi_{i,\delta}$ by applying formula ii) in Theorem 2 inductively. So we obtain $\phi(\xi)$ as a system of formal power series. By the definition of $\phi_{i,\delta}$, the fundamental equation $\phi(\eta(\xi)) = f(\phi(\xi))$ is satisfied formally.

We employ the method of majorant in order to prove the convergence of ϕ near the origin. Take a real number $a > 1$ such that

$$(3) \quad |\alpha^\delta - \alpha_i| \geq a^{|\delta|} - a$$

for any multi-index δ and $i = 1, \dots, n$. Note that $\Gamma(\phi, \delta, d)$ are polynomials in $\phi_{i,r}$'s with coefficients all positive.

For a positive real number r , let

$$M(r) = \text{MAX}_{|x_i| \leq r} \left(\text{MAX}_{j=1, \dots, n} (|f_j(x) - \alpha_j x_j|) \right)$$

where x range over a neighborhood of the origin in n -dimensional complex space C^n and f_j are regarded as extended to a neighborhood of the origin in C^n . Then we have

$$\lim_{r \rightarrow 0} M(r) = 0 \quad \text{and} \quad \lim_{r \rightarrow 0} \frac{M(r)}{r} = 0.$$

Take r sufficiently small so that $a > nM(r)/r$ holds. Let

$$F_i(x) = ax_i + \frac{a(x_1 + x_2 + \dots + x_n)^2}{nr - n(x_1 + x_2 + \dots + x_n)}.$$

Function $F_i(x) - ax_i$ is a majorant of $f_i(x) - \alpha_i x_i$, i.e., if we write

$$F_i(x) = ax_i + \sum_{|d| \geq 2} F_{i,d} x^d,$$

we have

$$(4) \quad |f_{i,d}| \leq F_{i,d}$$

for all i and d with $|d| \geq 2$.

Define $F: R^n \rightarrow R^n$ by $F(x) = (F_1(x), \dots, F_n(x))$. Let $\mathcal{E} = (\xi_1, \dots, \xi_n)$ denote the coordinate on $T_0 R^n$ associated with basis $\partial/\partial x_1, \dots, \partial/\partial x_n$.

Let $\Delta = (\Delta_1, \Delta_2, \dots, \Delta_n)$ be multi-index. Notations $|\Delta|, \Delta(i), \mathcal{E}^\Delta$ are defined similarly as for δ and ξ . Let $\Phi: T_0 R^n \rightarrow R^n$ be the formal power series $\Phi(\mathcal{E}) = (\Phi_1(\mathcal{E}), \dots, \Phi_n(\mathcal{E}))$ derived from the fundamental equation

$$(5) \quad \Phi(a\mathcal{E}) = F(\Phi(\mathcal{E}))$$

by assigning

$$(6) \quad \Phi_{i,\Delta(j)} = 1$$

for $i, j = 1, \dots, n$, and by applying the formula

$$(7) \quad \Phi_{i,d} = \frac{1}{a^{|\Delta|} - a} \left(\sum_{d \in \mathcal{I}(\mathbb{Z}, |\Delta|)} F_{i,d}(\Phi, \Delta, d) \right).$$

For each $\delta = (\delta_1, \dots, \delta_k)$, let $\Delta(\delta) = (\delta_1, \dots, \delta_k, 0, \dots, 0)$. We have

$$|\phi_{i,\delta}| \leq \Phi_{i,\Delta(\delta)}$$

inductively. So if we find Φ satisfying (5) and (6) with positive radius of convergence then ϕ also converges near the origin. The domain of definition can be extended to the total space E^n by virtue of fundamental equation

$$\phi(\eta(\xi)) = f(\phi(\xi)),$$

since η is an expanding linear automorphism.

The convergence of Φ near the origin is verified as follows. We claim that

$$\Phi_i(\mathcal{E}) = \frac{r(a-1)(\xi_1 + \dots + \xi_n)}{r(a-1) - n(\xi_1 + \dots + \xi_n)}$$

for $\mathcal{E} = (\xi_1, \dots, \xi_n)$. Note that formal power series is uniquely determined by (6) and (7), hence by (6) and (5). For each $i = 1, \dots, n$ and $j = 1, \dots, n$, we have

$$\frac{\partial \Phi_i}{\partial \xi_j}(0) = 1.$$

Let $\Sigma = \xi_1 + \dots + \xi_n$ and $S(\mathcal{E}) = \Phi_1(\mathcal{E}) + \dots + \Phi_n(\mathcal{E}) = n\Phi_i(\mathcal{E})$. Then,

$$\Phi_i(\mathcal{E}) = \frac{r(a-1)\Sigma}{r(a-1) - n\Sigma} \quad \text{and} \quad \Phi_i(a\mathcal{E}) = \frac{ra(a-1)\Sigma}{r(a-1) - na\Sigma}.$$

On the other hand, we have

$$\begin{aligned} F_i(\Phi(\mathcal{E})) &= a\Phi_i(\mathcal{E}) + \frac{a(S(\mathcal{E}))^2}{nr - nS(\mathcal{E})} \\ &= \frac{arS(\mathcal{E})}{n(r - S(\mathcal{E}))} = \frac{ra(a-1)\Sigma}{r(a-1) - na\Sigma} = \Phi_i(a\mathcal{E}), \end{aligned}$$

so that

$$\Phi(a\mathcal{E}) = F(\Phi(\mathcal{E})),$$

which completes the proof of Theorem 2.

Taking in considerations that the image of a neighborhood of the origin O in E^u is mapped onto a local unstable manifold of O in R^n , Theorem 1 is easily verified.

§ 3. **Remarks.** If f is a real analytic map defined on an open set U in R^n containing the origin and if the image $f(U)$ is included in U , Theorems 1 and 2 hold.

If we replace f by a holomorphic map $f: C^n \rightarrow C^n$ defined globally on C^n , similar results hold. In this case the obtained map ϕ is entire on E^u .

If f is a holomorphic map defined on an open set U in C^n containing the origin and if the image $f(U)$ is included in U , ϕ is again entire on E^u .

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