54. On Topological Characterizations of Complex Projective Spaces and Affine Linear Spaces

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In § 1 we present several conjectures. In § 2 we give partial answers to them. In § 3 we discuss remaining problems.

§ 1. **Conjectures.** Conjecture (A_n). *Let U be a complex manifold of dimension n with the homotopy type of a point. Suppose that there is a Kähler smooth compactification M of U such that $D=M-U$ is a smooth divisor on M. Then U is isomorphic to an affine linear space A^n.*

Remark 1. The smoothness of D is the essential assumption. Without it, U need not be A^n (see [12]).

In § 2 we reduce (A_n) to the following

Conjecture (B_n). *Let M be a compact complex manifold with $\dim M=n$ and let D be a smooth ample divisor on M. Suppose that the natural homomorphism $H_p(D;Z)\to H_p(M;Z)$ is bijective for $0\leqq p \leqq 2n-2$. Then $M\cong P^n$ and D is a hyperplane section on it.*

Remark 2. An affirmative answer to (B_n) would solve the question of [5] (4.15) and give a sharpened form of Proposition V in [13]. See also § 2, Corollary 3.

In § 2 we reduce (B_n) to the following

Conjecture (C_n). *Let M be a projective complex manifold such that the cohomology ring $H^{\cdot}(M;Z)$ is isomorphic to $H^{\cdot}(P^n;Z) \cong Z[x]/(x^{n+1})$. Suppose further that $c_1(M)$ is positive. Then $M\cong P^n$.*

Remark 3. It is well known that any projective manifold homeomorphic to P^n is holomorphically isomorphic to P^n, provided that c_1 is positive. Moreover, the positivity assumption on c_1 is not necessary if n is odd (see [8] and [11]). The proof depends on the theory of Pontrjagin classes.

Remark 4. (C_n) would not be true without the assumption on the ring structure. Indeed, any odd dimensional hyperquadric has a cohomology group isomorphic to that of P^n.

§ 2. **Partial answers.** Theorem 1. *Conjecture C_n is true for $n\leqq 5$.*

We give an outline of our proof for the case $n=5$. In view of the isomorphism $H^{\cdot}(M;Z)\cong H^{\cdot}(P^n;Z)$, we regard the Chern classes

$\{c_i\}$ of M as integers. First we have $c_5=6$. c_1 is a positive integer by assumption. Moreover, $M \cong \boldsymbol{P}^5$ if $c_1 \geqq 6$ (see [10] or [4]). So we may assume $5 \geqq c_1 \geqq 1$.

Let L be the ample generator of $\mathrm{Pic}(M) \cong H^2(M; \boldsymbol{Z}) \cong \boldsymbol{Z}$. Write down explicitly the Riemann-Roch-Hirzebruch formulae (see [7]) for $\chi(M, \mathcal{O}_M[tL])$, $\chi(M, \Omega[tL])$ and $\chi(M, \Theta[tL])$, where Ω is the sheaf of holomorphic 1-forms and Θ is the sheaf of holomorphic vector fields on M. Since $\chi(M, tL)$ is an integer for any $t \in \boldsymbol{Z}$, we infer that c_1 is even. Hence we should consider the cases $c_1=2$ or 4.

In case $c_1=2$, the equations among $\{c_j\}$ derived from $\chi(M, \mathcal{O}_M)=1$ and $\chi(M, \Omega)=-1$ imply that $c_4=45$ and $3c_2^2-4c_2+2c_3=765$. On the other hand, $\chi(M, L) \in \boldsymbol{Z}$ and $\chi(M, \Theta[-L]) \in \boldsymbol{Z}$ imply $-1 \equiv c_2 \equiv c_3 + 7 \bmod 12$. This is not consistent with the above equation.

In case $c_1=4$, we have $\chi(M, \mathcal{O}_M[tL])=0$ for $t=-1, -2$ and -3. Using this, we can derive a contradiction by a similar method as above.

The proofs in the cases $n \leqq 4$ are similar and simpler. Q.E.D.

Remark 5. In case $c_1=n-1$, we can also use the theory of Del Pezzo manifolds in order to derive a contradiction (cf. [3] or [6]).

Theorem 2. *Suppose that (M, D) satisfies the hypothesis of Conjecture B_n. Then $H^{\cdot}(M; \boldsymbol{Z}) \cong H^{\cdot}(\boldsymbol{P}^n; \boldsymbol{Z})$ and $H^{\cdot}(D; \boldsymbol{Z}) \cong H^{\cdot}(\boldsymbol{P}^{n-1}; \boldsymbol{Z})$ as graded rings. Moreover, $[D]$ generates* $\mathrm{Pic}(M)$ *and both $c_1(M)$ and $c_1(D)$ are positive.*

Proof (mostly due to Sommese [13], Proposition V). Let $f: D \to M$ be the inclusion. $f^*: H^p(M; \boldsymbol{Z}) \to H^p(D; \boldsymbol{Z})$ is bijective for $0 \leqq p \leqq 2n-2$ since $f_*: H_p(D; \boldsymbol{Z}) \to H_p(M; \boldsymbol{Z})$ is so. We have $H_p(D; \boldsymbol{Z}) \cong H^{2n-2-p}(D; \boldsymbol{Z})$ and $H_p(M; \boldsymbol{Z}) \cong H^{2n-p}(M; \boldsymbol{Z})$ by the Poincaré duality. Hence f_* induces a bijection $f': H^q(D; \boldsymbol{Z}) \to H^{q+2}(M; \boldsymbol{Z})$ for $0 \leqq q \leqq 2n-2$. Putting $\alpha = c_1([D]) \in H^2(M; \boldsymbol{Z})$, we see $f' \circ f^*(x) = x \wedge \alpha$ for any $x \in H^{\cdot}(M; \boldsymbol{Z})$. So the bijectivity of f' and f^* implies that α^k generates $H^{2k}(M; \boldsymbol{Z}) \cong \boldsymbol{Z}$ for any $0 \leqq k \leqq n$. In particular we have $\alpha^n = 1$ in $H^{2n}(M)$.

Assume that $b_1(M) > 0$. Then the Albanese mapping $\pi: M \to \mathrm{Alb}(M)$ is non-trivial. On the other hand, $H^2(M; \boldsymbol{Z}) \cong \boldsymbol{Z}$ implies that $H^{2,0}(M) = H^0(M, \Omega^2) = 0$. Hence $\pi(M)$ is a curve, since otherwise $\pi^*\psi \neq 0$ for some holomorphic 2-form ψ on $\mathrm{Alb}(M)$ (see [14], p. 116). A fiber of π is an effective divisor on M which is not ample. This is impossible since $H^2(M; \boldsymbol{Z}) \cong \boldsymbol{Z}$. This contradiction proves $b_1(M)=0$.

Now we have $H^1(M; \boldsymbol{Z})=0$ since it is torsion free. In view of the bijections f' and f^*, we infer that $H^p(M; \boldsymbol{Z})=0$ for any odd p. Thus we obtain a ring isomorphism $H^{\cdot}(M; \boldsymbol{Z}) \cong H^{\cdot}(\boldsymbol{P}^n; \boldsymbol{Z})$. Moreover, $\mathrm{Pic}(M) \cong H^2(M; \boldsymbol{Z})$ because $h^{0,1}(M) = h^{0,2}(M) = 0$, and $\mathrm{Pic}(M)$ is generated by $[D]$. Using f^*, we see $H^{\cdot}(D; \boldsymbol{Z}) \cong H^{\cdot}(\boldsymbol{P}^{n-1}; \boldsymbol{Z})$.

Assume that $c_1(M) \leqq 0$. Then the canonical bundle K of M is a non-negative multiple of $[D] \in \mathrm{Pic}(M)$. But $h^0(M, K) = h^{n,0}(M) = 0$ since

$H^n(M;Z)\cong H^n(P^n;Z)$. This contradiction proves that $c_1(M)>0$. Hence $c_1(D)=c_1(M)-1\geqq 0$. $h^{n-1,0}(D)=0$ implies $c_1(D)\neq 0$. So we have $c_1(D)>0$. Q.E.D.

Corollary 1. *(C_n) implies (B_n) and (B_{n+1}).*

Corollary 2. *Conjecture B_n is true for $n\leqq 6$.*

Corollary 3. *Let $f: M\to S$ be a surjective holomorphic mapping between compact complex manifolds and let A be a smooth ample divisor on M such that the restriction $f_A: A\to S$ of f is everywhere of maximal rank. Suppose that $\dim M\leqq 2\dim S+1$ and $\dim M\leqq \dim S+6$. Then both f and f_A are fiber bundles with fibers being isomorphic to projective spaces.*

For a proof, use [13] Proposition V and [5], (4.9).

Theorem 3. *Let U, M and D be as in Conjecture A_n. Then (M, D) satisfies the hypothesis of Conjecture B_n.*

Proof. By the Lefschetz duality we have $H_p(M,D;Z)\cong H^{2n-p}(U;Z)=0$ for $p\leqq 2n-1$. Hence, the long homology exact sequence proves $H_p(D;Z)\cong H_p(M;Z)$ for $p\leqq 2n-2$. This implies, as in Theorem 2, that $H^2(M;Z)$ is generated by $c_1([D])$. On the other hand M is Kähler. Therefore D is ample since any Kähler class of M is a positive multiple of $c_1([D])$.

Corollary 4. *(B_n) implies (A_n).*

Corollary 5. *Conjecture A_n is true for $n\leqq 6$.*

Remark. Actually, we used only $H^q(U;Z)=0$, and not $\pi_i(U)=0$.

§3. Comments. (3.1) It is doubtful if the computational method as in Theorem 1 works in higher dimensional cases. However, this method might work in (B_n) even though it doesn't in (C_{n-1}). So the first non-solved case is (B_7).

(3.2) Without the assumption $c_1(M)>0$, (C_n) might not be true. But, so far as I know, there is no counter-example. I suspect that there will be only few types of such manifolds. In particular, n might be necessarily even.

(3.3) Combining the results of Yau [15] and Kobayashi [9], we infer that $c_1(M)>0$ implies $\pi_1(M)=0$. So we may assume that M is simply connected in (A_n), (B_n) and (C_n). Hence the rational homotopy type of M is same to that of P^n (cf. [2]). Does this imply that M is homeomorphic to P^n? If yes, then our conjectures are solved.

(3.4) In positive characteristic cases we can formulate analogues of (A_n), (B_n) and (C_n) in terms of Chow rings and some cohomology theory. However, I have no answer except trivial cases. One of the main difficulties is the lack of vanishing theorems of Kodaira type.

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