# 48. Branching of Singularities for Degenerate Hyperbolic Operators and Stokes Phenomena

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In terms of Fourier integral operators we give an explicit representation of solutions of the Cauchy problem for a certain class of degenerate hyperbolic equations, and determine precisely whether or not the solutions possess branching singularities. Our results reveal a close connection between branching of singularities and Stokes phenomena.

Alinhac [1] and Taniguchi-Tozaki [5] studied the problem of branching singularities for a special class of operators $\frac{\partial^2}{\partial t^2}-t^{2l}\frac{\partial^2}{\partial x^2}+$ (lower order terms) in $\boldsymbol{R}_t\times\boldsymbol{R}_x$. We study the same problem for the following type of higher order operators $P(t, D_t, D_x)$ in $\boldsymbol{R}_t\times\boldsymbol{R}^n_x$:

$$P(t,D_t,D_x)=\sum_{i=0}^{m}P_{m-i}(t,D_t,D_x),$$

$$P_{m-i}(t,D_t,D_x)=\sum_{j=0}^{m-i}t^{jl-i}p_{i,j}(D_x)D_t^{m-i-j},\qquad 0\leqq i\leqq m,$$

$$P_m(t,\tau,\xi)=\prod_{i=1}^{m}(\tau-t^l\lambda_i(\xi)),$$

where $D_t=\frac{1}{\sqrt{-1}}\frac{\partial}{\partial t}$, $D_x=(D_{x_1},\cdots,D_{x_n})=\left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_1},\cdots,\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_n}\right)$, $l\in\boldsymbol{N}$, $p_{i,j}(\xi)\in C^\infty(\boldsymbol{R}^n_\xi)$ are homogeneous of degree $j$ with respect to $\xi$, $p_{i,j}(\xi)\equiv 0$ if $jl-i<0$ and $\lambda_i(\xi)\in C^\infty(\boldsymbol{R}^n_\xi\backslash 0, \boldsymbol{R}\backslash 0)$ are distinct. Here $C^\infty(\boldsymbol{R}^n_\xi\backslash 0,\boldsymbol{R}\backslash 0)$ denotes the set of all $\boldsymbol{R}\backslash 0$-valued $C^\infty$ functions defined in $\boldsymbol{R}^n_\xi\backslash 0$.

**Lemma 1.** *If we introduce a new independent variable* $z=\frac{t^{l+1}}{l+1}|\xi|$, *then the ordinary differential operator* $P\left(t,\frac{1}{\sqrt{-1}}\frac{d}{dt},\xi\right)$ *is written as*

$$P\left(t,\frac{1}{\sqrt{-1}}\frac{d}{dt},\xi\right)=\sqrt{-1}^{-m}(l+1)^{lm/(l+1)}|\xi|^{m/(l+1)}z^{-m/(l+1)}L\left(z,\frac{d}{dz},|\xi|^{-1}\xi\right)$$

*for* $\xi\in\boldsymbol{R}^n_\xi\backslash 0$, *where*

$$L\left(z,\frac{d}{dz},\theta\right)=\sum_{i=0}^{m}\left(\sum_{j=0}^{i}a_{i,j}(\theta)z^j\right)z^{m-i}\frac{d^{m-i}}{dz^{m-i}},$$

$a_{i,j}(\theta)=\sum_{j\leqq k\leqq h\leqq i}\sqrt{-1}^k(l+1)^{j-h}\alpha_1(m-k,m-h)\alpha_2(m-h,m-i)p_{k-j,j}(\theta)$ *for* $\theta\in S^{n-1}_\theta$ *and*

$$\begin{cases} \alpha_1(i,i)=1 \quad \text{for } i \geqq 0, \quad \alpha_1(i,0)=0 \quad \text{for } i>0, \\ \alpha_1(i+1,j)=\alpha_1(i,j-1)-i\alpha_1(i,j) \quad \text{for } 1 \leqq j \leqq i, \end{cases}$$
$$\begin{cases} \alpha_2(i,i)=1 \quad \text{for } i \geqq 0, \quad \alpha_2(i,0)=0 \quad \text{for } i>0, \\ \alpha_2(i+1,j)=\alpha_2(i,j-1)+j\alpha_2(i,j) \quad \text{for } 1 \leqq j \leqq i. \end{cases}$$

The original form of this lemma will be found in Nakamura-Uryu [3].

For simplicity, we assume $m<l+2$ (this assumption is removable). Then, as is well-known (cf. Okubo-Kohno [4] and Wasow [6]), the equation $L\left(z, \frac{d}{dz}, \theta\right)V(z)=0$ in $C_z$, with a parameter $\theta \in S_\theta^{n-1}$, has a fundamental set of solutions

$$V_i(z,\theta)=z^{(i-1)/(l+1)}\sum_{\nu=0}^{\infty} g_i(\nu,\theta)z^\nu, \qquad 1 \leqq i \leqq m$$

near the regular singular point $z=0$ and a set of formally independent formal solutions

$$W_j(z,\theta)=\exp(\sqrt{-1}\lambda_j(\theta)z)z^{\mu_j(\theta)}\sum_{\nu=0}^{\infty} h_j(\nu,\theta)z^{-\nu}, \qquad 1 \leqq j \leqq m$$

near the irregular singular point $z=\infty$, where

$$\mu_j(\theta)=\frac{\sum_{k=2}^{m}\lambda_j^{m-k}(\theta)p_{1,k-1}(\theta)}{(l+1)\prod_{\substack{k=1\\k\neq j}}^{m}(\lambda_j(\theta)-\lambda_k(\theta))}$$

and $g_i(\nu,\theta)$, $h_j(\nu,\theta)$ are determined as follows. $g_i(0,\theta)=1$ and

$$\begin{cases} g_i(1,\theta)\zeta_0\left(1+\frac{i-1}{l+1},\theta\right)+g_i(0,\theta)\zeta_1\left(\frac{i-1}{l+1},\theta\right)=0 \\ g_i(\nu,\theta)\zeta_0\left(\nu+\frac{i-1}{l+1},\theta\right)+\sum_{k=1}^{\min(m,\nu)} g_i(\nu-k,\theta)\zeta_k\left(\nu-k+\frac{i-1}{l+1},\theta\right)=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nu \geqq 2, \end{cases}$$

where $\zeta_j(\rho,\theta)=\sum_{i=j}^{m} a_{i,j}(\theta)[\rho]_{m-i}$ and $[\rho]_k=\rho(\rho-1)\cdots(\rho-k+1)$. $h_j(0,\theta)=1$, $h_j(\nu,\theta)=0$ for $\nu<0$ and

$$\begin{cases} h_j^k(\nu,\theta)=\sqrt{-1}\lambda_j(\theta)h_j^{k-1}(\nu,\theta)+(\mu_j(\theta)-\nu+1)h_j^{k-1}(\nu-1,\theta), \qquad 1 \leqq k \leqq m \\ h_j^m(\nu,\theta)=\sum_{p=1}^{m}\sum_{q=0}^{\min(p,\nu)} a_{p,p-q}(\theta)h_j^{m-p}(\nu-q,\theta), \end{cases}$$

where

$$W_j^0(z,\theta)=\exp(\sqrt{-1}\lambda_j(\theta)z)z^{\mu_j(\theta)}\sum_{\nu=0}^{\infty} h_j^0(\nu,\theta)z^{-\nu}, \qquad h_j^0(\nu,\theta)=h_j(\nu,\theta).$$

$$W_j^k(z,\theta)=\frac{d}{dz}W_j^{k-1}(z,\theta)=\exp(\sqrt{-1}\lambda_j(\theta)z)z^{\mu_j(\theta)}\sum_{\nu=0}^{\infty} h_j^k(\nu,\theta)z^{-\nu}.$$

We now consider two sectors $S_{+1}=\left\{z : |\arg z| \leqq \frac{\pi}{2}-\varepsilon\right\}$, $0<\varepsilon=\text{const}<\frac{\pi}{2}$, and $S_{-1}=\{z\exp(\sqrt{-1}(l+1)\pi) : z \in S_{+1}\}$. By the asymptotic existence theorem (cf. Okubo-Kohno [4] and Wasow [6]), we can show the

$C^\infty$-smoothness of Stokes multipliers with respect to the parameter $\theta \in S_\theta^{n-1}$.

**Lemma 2.** *There exist functions $T_\delta^{i,j}(\theta) \in C^\infty(S_\theta^{n-1})$, $\delta = \pm 1$, $1 \leqq i$, $j \leqq m$, such that*

$$V_i(z,\theta) \sim \sum_{j=1}^{m} T_\delta^{i,j}(\theta) W_j(z,\theta) \qquad \text{in } S_\delta$$

*as $z \to \infty$, where "$\sim$" stands for the asymptotic equality.*

If we set

$$H_j^k(z,\theta) = \frac{\sum_{i=1}^{m} \tilde{T}_\delta^{i,j}(\theta) \frac{d^k}{dz^k} V_i(z,\theta)}{\det\left(T_\delta^{p,q}(\theta) : \begin{matrix} p \downarrow 1 \cdots m \\ q \to 1 \cdots m \end{matrix}\right) \exp(\sqrt{-1}\lambda_j(\theta)) z^{\mu_j(\theta)}}$$

for $z \in S_\delta$, $\delta = \pm 1$, where $\tilde{T}_\delta^{i,j}(\theta)$ is the $(i,j)$ cofactor of the matrix $\left(T_\delta^{p,q}(\theta) : \begin{matrix} p \downarrow 1 \cdots m \\ q \to 1 \cdots m \end{matrix}\right)$, then we have the following

**Lemma 3.**

$$H_j^k(z,\theta) \sim \sum_{\nu=0}^{\infty} h_j^k(\nu,\theta) z^{-\nu} \qquad \text{in } S_{+1} \cup S_{-1}$$

*as $z \to \infty$; this gives*

$$\left(H_j^k(z,\theta) : \begin{matrix} k \downarrow 0 \cdots m-1 \\ j \to 1 \cdots m \end{matrix}\right) \sim \left(\sqrt{-1}^k \lambda_j^k(\theta) : \begin{matrix} k \downarrow 0 \cdots m-1 \\ j \to 1 \cdots m \end{matrix}\right) + 0\left(\frac{1}{z}\right)$$
$$\text{in } S_{+1} \cup S_{-1},$$

*where $\lambda_j^k(\theta) = (\lambda_j(\theta))^k$.*

$\mathcal{E}'(\boldsymbol{R}_x^n)$ denotes the set of all distributions in $\boldsymbol{R}_x^n$ with compact supports. $\hat{u}(\xi) = \mathcal{F}_x[u(x)](\xi)$ is the Fourier transformation and $WF(u)$ is the wave front set of $u(x)$. $\mathcal{O}_M(\boldsymbol{R}_{t,s}^2 \times \boldsymbol{R}_\xi^n)$ denotes the set of all slowly increasing functions in $\boldsymbol{R}_{t,s}^2 \times \boldsymbol{R}_\xi^n$. $\overline{\mathcal{F}}_\xi[U(t,s,\xi)](x)$ is the inverse Fourier transformation of $U(t,s,\xi)$ with respect to $\xi$.

**Theorem.** *For any $u_k(x) \in \mathcal{E}'(\boldsymbol{R}_x^n)$, $0 \leqq k \leqq m-1$, there exists a function $U(t,s,\xi) \in \mathcal{O}_M(\boldsymbol{R}_{t,s}^2 \times \boldsymbol{R}_\xi^n)$ such that*

$$u(t,s,x) = \overline{\mathcal{F}}_\xi[U(t,s,\xi)](x) \left(= (2\pi)^{-n} \int \exp(\sqrt{-1}x\cdot\xi) U(t,s,\xi) d\xi\right)$$

*is a solution of the Cauchy problem*

$$\begin{cases} P(t,D_t,D_x)u(t,s,x) = 0 & \text{in } \boldsymbol{R}_t \times \boldsymbol{R}_x^n \\ \dfrac{\partial_k}{\partial t^k} u(t,s,x)|_{t=s} = u_k(x), & 0 \leqq k \leqq m-1 \end{cases}$$

*and $U(t,s,\xi)$ admits of the expression*

$$U(t,s,\xi) = \sum_{k=0}^{m-1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\sum_{p=1}^{m} T_{\operatorname{sgn} t}^{p,i}(|\xi|^{-1}\xi) \tilde{T}_{\operatorname{sgn} s}^{p,j}(|\xi|^{-1}\xi)\right)$$
$$\times \exp\left(\sqrt{-1}\left(\frac{t^{l+1}}{l+1}\lambda_i(\xi) - \frac{s^{l+1}}{l+1}\lambda_j(\xi)\right)\right)\left(\frac{t^{l+1}}{l+1}|\xi|\right)^{\mu_i(|\xi|^{-1}\xi)}$$
$$\times \left(\frac{s^{l+1}}{l+1}|\xi|\right)^{-\mu_j(|\xi|^{-1}\xi)} \left(\prod_{q=0}^{m-1} \frac{1}{q!}\right) s^{m(m-1)/s}$$
$$\times J_i(t,\xi) \tilde{J}_j^k(s,\xi) \hat{u}_k(\xi)$$

*if* $t\neq 0$, $s\neq 0$ *and* $|\xi|\neq 0$, *where*

$$J_j^k(t,\xi)=\sum_{0\leqq p\leqq q\leqq k}(l+1)^{q-p}\alpha_1(k,q)\alpha_2(q,p)t^{p(l+1)-k}|\xi|^p H_j^p\Big(\frac{t^{l+1}}{l+1}|\xi|,|\xi|^{-1}\xi\Big)$$

*and* $\tilde{J}_j^k(s,\xi)$ *are the* $(k,j)$ *cofactors of the matrix* $\Big(J_q^p(s,\xi): \begin{smallmatrix} p\downarrow 0\cdots m-1 \\ q\rightarrow 1\cdots m\end{smallmatrix}\Big)$.

When either $t$ or $s=0$ the function $U(t,s,\xi)$ admits of a simpler expression than the above one. We can prove a similar theorem for a certain class of degenerate hyperbolic systems.

Generally, the matrix $\Big(\sum_{p=1}^m T_{+\delta}^{p,i}(\theta)\tilde{T}_{-\delta}^{p,j}(\theta): \begin{smallmatrix} j\downarrow 1\cdots m \\ i\rightarrow 1\cdots m\end{smallmatrix}\Big)$ is not diagonal; this fact is called the Stokes phenomenon. We can show that a solution possesses branching singularities for some singular Cauchy data if and only if the Stokes phenomenon occurs. In fact, "only if" part follows immediately from Theorem, and the following corollary proves the truth of "if" part.

**Corollary.** *If* $\sum_{p=1}^m T_{\mathrm{sgn}\,t_0}^{p,i}(\theta^0)\tilde{T}_{\mathrm{sgn}\,s_0}^{p,j}(\theta^0)\neq 0$, $WF(u_k)=\{(x^0,\rho\theta^0):\rho>0\}$, $\bigcup_{q=0, q\neq k}^{m-1} WF(u_q)=\emptyset$ *and* $\tilde{\lambda}_j^k(\theta^0)\neq 0$ *for some* $0\leqq k\leqq m-1$, $1\leqq i$, $j\leqq m$, $t_0\neq 0$, $s_0\neq 0$, $x^0\in \boldsymbol{R}_x^n$ *and* $\theta^0\in S_\theta^{n-1}$, *then* $u(t_0,s_0,x)$ *possesses the singularity caused by the phase function* $x\cdot\xi+\dfrac{t_0^{l+1}}{l+1}\lambda_i(\xi)-\dfrac{s_0^{l+1}}{l+1}\lambda_j(\xi)$, *where* $\tilde{\lambda}_j^k(\theta^0)$ *is the* $(i,j)$ *cofactor of the matrix* $\Big(\lambda_q^p(\theta^0): \begin{smallmatrix} p\downarrow 0\cdots m-1 \\ q\rightarrow 1\cdots m\end{smallmatrix}\Big)$.

Clearly, $\tilde{\lambda}_j^k(|\xi^0|^{-1}\xi^0)\neq 0$ if $k=0$ or $m-1$. It is to be noted that the phase functions $x\cdot\xi+\dfrac{t^{l+1}}{l+1}\lambda_i(\xi)-\dfrac{s^{l+1}}{l+1}\lambda_j(\xi)$ are the multi-phase functions introduced by Kumano-go-Taniguchi-Tozaki [2].

## References


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