44. On Formal Analytic Poincaré Lemma*)

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1. Introduction. Let X be a complex manifold and Y an analytic subset of X. Let $\Omega_X^{\cdot}$ be the complex of sheaves of germs of holomorphic forms on X and $\hat{\Omega}_X^{\cdot}$ the formal completion of $\Omega_X^{\cdot}$ along Y (cf. [4]). Then the formal analytic Poincaré lemma says that $\hat{\Omega}_X^{\cdot}$ gives a resolution of C_Y with respect to the natural augmentation $C_Y \to \hat{\Omega}_X^{\cdot}$. This was first shown by Hartshorne [4] and Sasakura [5] independently. Actually, Sasakura obtained a stronger result using his theory of stratifying analytic sets and of cohomology with growth conditions [5]. In the present note we shall give a simple alternative proof of his result using resolution, based on the idea of Bloom (cf. [2, 3.1]).

2. Statement of the result. Let $U = X - Y$ and $j: U \to X$ be the inclusion. Let I be any coherent sheaf of ideals of O_X with supp $O_X/I = Y$ where supp denotes the support. We call an open subset V of X *good with respect to* Y if V is Stein, its closure $\bar{V}$ is a Stein compact, and if the restriction map $j^*: H^i(V, C) \to H^i(Y \cap V, C)$ are isomorphic for all i, or equivalently, $H^i_{\Psi}(V - V \cap Y, C) = 0$ for all i, where Ψ is the family of supports consisting of closed subsets of V which are contained in $V - V \cap Y$. In what follows for a rational number r we denote by $[r]$ the largest integer which is not greater than r, and then we write $[r]_+ = \max([r], 0)$.

Theorem. *Let V be an open subset of X which is good with respect to Y. Then there exist rational numbers c_1, c_2 with $c_1 > 0$ such that if we put $c(m) = [c_1 m - c_2]_+$ for any integer m, then the following hold true*: 1) *For every $p > 0$ and $\varphi \in \Gamma(V, I^m \Omega_X^p)$ with $d\varphi = 0$ we can find a $\psi \in \Gamma(V, I^{c(m)} \Omega_X^{p-1})$ such that $\varphi = d\psi$.* 2) *Suppose further that V is contractible. Then for every $p \geq 0$ and every $\varphi \in \Gamma(V, \Omega_X^p)$ with $d\varphi \in \Gamma(V, I^m \Omega_X^{p+1})$ we can find a $\psi \in \Gamma(V, \Omega_X^{p-1})$ such that $\varphi - d\psi \in \Gamma(V, I^{c(m)} \Omega_X^p)$, where $\Omega_X^{-1} = C_X$ and $d: \Omega_X^{-1} \to O_X$ is the natural inclusion.*

The formal Poincaré analytic lemma mentioned above follows from 2) of the above theorem together with the following:

Remark. For each $y \in Y$ there exists a fundamental system $\{V\}$ of contractible open neighborhoods V of y in X which are good with respect to Y. In fact, since the problem is local, we may assume that $X = C^n := C^n(z_1, \cdots, z_n)$ where $n = \dim X$. Let $r = \sum_{i=1}^n |z_i|^2$ and $D_\varepsilon = \{r < \varepsilon\}$

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for $\varepsilon>0$. Then it suffices to take $\{V\}=\{D_\varepsilon\}_{\varepsilon<\varepsilon_0}$ for some sufficiently small $\varepsilon_0>0$. Indeed, then we have only to see that $D_\varepsilon\cap Y$, $\varepsilon<\varepsilon_0$, is contractible, which is a consequence of Thom-Mather's theory (cf. [3, Chap. II, Theorem 5.4]).

3. **Proof of Theorem.** 2) follows from 1) as follows. Put $\eta=d\varphi$. Then $d\eta=0$ so that by 1) there exists a $\xi\in\Gamma(V, I^{c(m)}\Omega_X^p)$ such that $\eta=d\xi$. Then $d(\varphi-\xi)=0$, so we get a $\psi\in\Gamma(V,\Omega_X^{p-1})$ such that $d\psi=\varphi-\xi$ since V is Stein and contractible. Hence $\varphi-d\psi=\xi\in\Gamma(V, I^{c(m)}\Omega_X^p)$.

Next we show 1). Take by Hironaka a proper bimeromorphic morphism $f:\tilde{X}\to X$ with $\tilde{X}$ nonsingular and $\tilde{Y}:=f^{-1}(Y)$ a divisor with only normal crossings in $\tilde{X}$, such that $f|_{\tilde{X}-\tilde{Y}}:\tilde{X}-\tilde{Y}\to X-Y$ is isomorphic. We may further assume that on the closure $\bar{V}$ of V, f is obtained by blowing up of a coherent analytic sheaf I' of ideals of O_X with supp $O_X/I'=Y$.

We first consider the case where $I=I'$ and show that in this case 1) is true with $c_1=1$. We put $\tilde{I}=f^{-1}(I')\ (=I'O_{\tilde{X}})$. Then $\tilde{I}$ is f-very ample on $\bar{V}$. Let $\tilde{\varphi}=\varphi f$. Then $\tilde{\varphi}\in\Gamma(\tilde{V},\tilde{I}^m\Omega_{\tilde{X}}^p)$ and $d\tilde{\varphi}=0$, where $\tilde{V}=f^{-1}(V)$. Then the main point of our proof is to show the following:
(+) *There exists an integer m_0 such that once $m\geqq m_0$, then we can always find a $\tilde{\psi}\in\Gamma(\tilde{V},\tilde{I}^{m-p}\Omega_{\tilde{X}}^{p-1})$ such that $d\tilde{\psi}=\tilde{\varphi}$.*

First we define a subcomplex $K_m^\cdot$ of $\Omega_{\tilde{X}}^\cdot$ by

$$K_m^\cdot=\tilde{I}^{m+1}\Omega_{\tilde{X}}^\cdot+\tilde{I}^m d\tilde{I}\wedge\Omega_{\tilde{X}}^{\cdot-1}.$$

Then we put $\Omega_{\tilde{Y}(m)}^\cdot=\Omega_{\tilde{X}}^\cdot/K_m^\cdot$, where $\tilde{Y}_{(m)}$ is the complex subspace of $\tilde{X}$ defined by the ideal $\tilde{I}^{m+1}$. We then have the obvious exact sequence of complexes

$$(1)\qquad 0\to K_m^\cdot\to\Omega_{\tilde{X}}^\cdot\to\Omega_{\tilde{Y}(m)}^\cdot\to 0.$$

On the other hand, by Reiffen (cf. [2, 3.1]), for every $m\geqq 1$ (1) is a resolution of the following exact sequence of sheaves of $\boldsymbol{C}$-vector spaces

$$(2)\qquad 0\to\tilde{j}_!\boldsymbol{C}_{\tilde{U}}\to\boldsymbol{C}_{\tilde{X}}\to\boldsymbol{C}_{\tilde{Y}}\to 0$$

with respect to the natural augmentation from (2) to (1), where $\tilde{U}=\tilde{X}-\tilde{Y}$ and $\tilde{j}:\tilde{U}\to\tilde{X}$ is the inclusion. From (1) and (2) we get on $\tilde{V}=f^{-1}(V)$ the following commutative diagram of hypercohomology exact sequences

$$\begin{array}{ccccccc}\longrightarrow & \boldsymbol{H}^q(\tilde{V},K_m^\cdot) & \longrightarrow & \boldsymbol{H}^q(\tilde{V},\Omega_{\tilde{X}}^\cdot) & \longrightarrow & \boldsymbol{H}^q(\tilde{V},\Omega_{\tilde{Y}(m)}^\cdot) & \longrightarrow\\ & \uparrow & & \uparrow & & \uparrow & \\ \longrightarrow & H^q_{\tilde{\Psi}}(\tilde{V}-\tilde{V}\cap\tilde{Y},\boldsymbol{C}) & \longrightarrow & H^q(\tilde{V},\boldsymbol{C}) & \longrightarrow & H^q(\tilde{Y}\cap\tilde{V},\boldsymbol{C}) & \longrightarrow\end{array}$$

where the vertical arrows are isomorphic and $\tilde{\Psi}=f^{-1}(\Psi)$. Since V is good, $H^q_{\tilde{\Psi}}(\tilde{V}-\tilde{V}\cap\tilde{Y},\boldsymbol{C})\cong H^q_\Psi(V-Y\cap V,\boldsymbol{C})=0$. Now we consider for each m the spectral sequence of hypercohomology associated to the complex $K_m^\cdot$

$$E_1^{p,q}(m)=H^q(\tilde{V},K_m^p)\Rightarrow\boldsymbol{H}^{p+q}(\tilde{V},K_m^\cdot)\cong H^{p+q}_{\tilde{\Psi}}(\tilde{V}-\tilde{V}\cap\tilde{Y},\boldsymbol{C})=0.$$

Corresponding to the natural inclusions $K^{\cdot}_m \subseteqq K^{\cdot}_{m'}$, $m \geqq m'$, we have the natural maps of the spectral sequences

$$\alpha_r^{p,q}(m, m') : E_r^{p,q}(m) \to E_r^{p,q}(m'), \qquad m \geqq m'.$$

Then we shall prove the following: ($''$) *If m is sufficiently large, then $\alpha_r^{p,q}(m, m-1)$ are zero maps for all $r \geqq 1$ and $q \geqq 1$.* In fact, by the definition of K_m^p we obtain the exact sequence

$$0 \to \tilde{I}^{m+1}\Omega_{\tilde{X}}^p \to K_m^p \to K_m^p/\tilde{I}^{m+1}\Omega_{\tilde{X}}^p \to 0.$$

Since $\tilde{I}$ is f-very ample on $\bar{V}$ and V is Stein, using Leray spectral sequence for f we have $H^q(\tilde{V}, \tilde{I}^{m+1}\Omega_{\tilde{X}}^p)=0$ for all sufficiently large m and $q \geqq 1$. Hence we get the natural isomorphisms

$$H^q(\tilde{V}, K_m^p) \cong H^q(\tilde{V}, K_m^p/\tilde{I}^{m+1}\Omega_{\tilde{X}}^p)$$

for sufficiently large m and $q \geqq 1$. On the other hand, since the compositions $K_{m+1}^p \to K_m^p \to K_m^p/\tilde{I}^{m+1}\Omega_{\tilde{X}}^p$ are zero maps, from the above isomorphisms we get that the natural maps $H^q(\tilde{V}, K_{m+1}^p) \to H^q(\tilde{V}, K_m^p)$ are all zero maps. Namely $\alpha_1^{p,q}(m, m-1)$ are zero maps and *a fortiori* $\alpha_r^{p,q}(m, m-1)$ are zero maps for $r \geqq 2$ for sufficiently large m and all $q \geqq 1$, which proves ($''$). Using ($''$) we next prove the following:

Assertion. *$\alpha_2^{p,0}(m, m-p+1) : E_2^{p,0}(m) \to E_2^{p,0}(m-p+1)$ are zero maps for all sufficiently large m and all $p>0$.*

Proof. It is enough to prove the following assertion (*) by descending induction on i: (*) *$\alpha_i^{p,0}(m, m-p+i-1) : E_i^{p,0}(m) \to E_i^{p,0}(m-(p-i+1))$ are zero maps for $p \geqq i \geqq 2$.* (The case $i=2$ corresponds to the above assertion.) We shall denote by $d_i^{p,q}(m) : E_i^{p,q}(m) \to E_i^{p+i,q-i+1}(m)$ the differentials of the spectral sequences. Suppose first that $i=p$. Then we have the natural isomorphisms $E_p^{p,0}(m) \cong \cdots \cong E_\infty^{p,0}(m)$ and the natural inclusion $E_\infty^{p,0}(m) \subseteqq H^p(\tilde{V}, K^{\cdot}_m)$. Since $H^p(\tilde{V}, K^{\cdot}_m)=0$ as was remarked above, we have $E_p^{p,0}(m)=0$. Hence (*) is true in this case. Next suppose that (*) is true for some $i>2$, so that we have $\alpha_i^{p,0}(m, m-p+i-1)E_i^{p,0}(m)=0$, i.e.,

$$\alpha_{i-1}^{p,0}(m, m-p+i-1)E_{i-1}^{p,0}(m) \subseteqq d_{i-1}^{p-i+1,i-2}(m-p+i-1)(E_{i-1}^{p-i+1,i-2}(m-p+i-1))$$

From this it follows that

$$\begin{aligned}
&\alpha_{i-1}^{p,0}(m, m-p+i-2)E_{i-1}^{p,0}(m)\\
&\quad=\alpha_{i-1}^{p,0}(m-p+i-1, m-p+i-2)\alpha_{i-1}^{p,0}(m, m-p+i-1)E_{i-1}^{p,0}(m)\\
&\quad\subseteqq\alpha_{i-1}^{p,0}(m-p+i-1, m-p+i-2)(d_{i-1}^{p-i+1,i-2}(m-p+i-1)\\
&\qquad\qquad(E_{i-1}^{p-i+1,i-2}(m-p+i-1)))\\
&\quad=d_{i-1}^{p-i+2,i-2}(m-p+i-2)(\alpha_{i-1}^{p-i+1,i-2}(m-p+i-1, m-p+i-2)\\
&\qquad\qquad(E_{i-1}^{p-i+1,i-2}(m-p+i-1)))\\
&\quad=0,
\end{aligned}$$

where the last equality comes from ($''$) since $i-2>0$ and $m-p+i-1$ is sufficiently large if m is. This proves (*) for $i-1$ and hence completes the inductive proof of (*), and hence of the assertion also.

The above assertion is equivalent to saying that the natural maps $\alpha_p(m): H^p\Gamma(\tilde{V}, K_m^\cdot) \to H^p\Gamma(\tilde{V}, K_{m-p+1}^\cdot)$ induced by the inclusions $K_m^\cdot \subseteq K_{m-p+1}^\cdot$ are zero maps for all sufficiently large m and all $p>0$. Now coming back to our $\tilde{\varphi}$, let $\bar{\varphi} \in H^p\Gamma(\tilde{V}, K_{m-1}^\cdot)$ be the class defined by $\tilde{\varphi}$. Then if m is sufficiently large, $\alpha_p(m-1)\bar{\varphi}=0$ so that there exists a $\psi \in \Gamma(\tilde{V}, K_{m-p}^{p-1})$ such that $\tilde{\varphi}=d\tilde{\psi}$. On the other hand, since $K_{m-p}^\cdot \subseteq \tilde{I}^{m-p}\Omega_{\tilde{X}}^\cdot$, $\tilde{\psi} \in \Gamma(\tilde{V}, \tilde{I}^{m-p}\Omega_{\tilde{X}}^{p-1})$. This proves (+).

From (+) we shall deduce our conclusion as follows. First note that for any coherent analytic sheaf F on $\tilde{X}$ there exists an integer $d>0$ such that for each $k>0$ the natural map $I^k f_*(\tilde{I}^d F) \to f_*(\tilde{I}^{d+k}F)$ is isomorphic. In fact, noting that $\tilde{X} \cong \operatorname{Projan}(\bigoplus_{\nu\geq 0}^\infty I^\nu)$ over $\bar{V}$ [1] this follows from the corresponding algebraic result (cf. EGA III 2.3.2) by the comparison theorems in Bingener [1]. We apply this to $F=\Omega_{\tilde{X}}^q$, $0\leqq q\leqq \dim X$, and obtain an integer $d>0$ independent of q such that

$$f_*(\tilde{I}^{d+k}\Omega_{\tilde{X}}^q) \cong I^k f_*(\tilde{I}^d\Omega_{\tilde{X}}^q) \subseteq I^k f_*(\Omega_{\tilde{X}}^q) \cong I^k\Omega_X^q$$

and hence $\Gamma(\tilde{V}, \tilde{I}^{d+k}\Omega_{\tilde{X}}^q) \subseteq \Gamma(V, I^k\Omega_X^q)$. Therefore if $\psi \in \Gamma(V, I^{m-p-d}\Omega_X^{p-1})$ corresponds to the above $\tilde{\psi}$ by this inclusion we have $\varphi=d\psi$. Thus if we set $c_1=1$ and $c_2=p+d$ then 1) is true for all sufficiently large m. Then taking c_2 larger 1) holds for all $m>0$.

Next in the general case fix positive integers s and t such that $I^s \subseteq I'$ and $I'^t \subseteq I$ on $\bar{V}$, which is possible by Hilbert zero theorem. Then by what we have proved above (applied to I') we see readily that there exist an integer c and a $\psi \in \Gamma(V, I^n\Omega_X^{p-1})$ with $d\psi=\varphi$ where $n=[(m_1-c)/t]_+$, $m_1=[m/s]$. Since $n\geqq (m/ts)-(c/t)-4$, if we set $c_1=1/ts$ and $c_2=-((c/t)+4)$, $\psi \in \Gamma(V, I^{c(m)}\Omega_X^{p-1})$. Q.E.D.

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