43. On a Conjecture of S. Chowla and of S. Chowla and H. Walum. III

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Let $P_r(v)$ denote the periodic Bernoulli polynomial of degree $r: P_r(v)=B_r(\{v\})$, where $B_r(v)$ is the r-th Bernoulli polynomial, $\{v\}=v-[v]$ being the fractional part of v ($[v]$ is the greatest integer not exceeding v). For $a \in \boldsymbol{R}$ and $r \in \boldsymbol{N}$ we put

$$G_{a,r}(x)=\sum_{n\leqq\sqrt{x}} n^a P_r\left(\frac{x}{n}\right). \tag{1}$$

Then Chowla and Walum's conjecture is that there holds the estimate

$$G_{a,r}(x)=O(x^{a/2+1/4+\varepsilon}) \tag{2}$$

for every positive ε (cf. [3], [6]). The case $r=1$ is concerned with Dirichlet's divisor problem and presents a difficulty of the highest degree, and the case $r=2$ is called Chowla's conjecture [4], [6], which seems to be as deep as the divisor problem itself: For every positive ε and $\psi(v)=\{v\}-\frac{1}{2}$

$$G_{0,2}(x)=\sum_{n\leqq\sqrt{x}}\left\{\psi^2\left(\frac{x}{n}\right)-\frac{1}{12}\right\}=O(x^{1/4+\varepsilon}). \tag{3}$$

We have proved in [6] that a stronger version of (2) is true if $a\geqq\frac{1}{2}$ and $r\geqq 2$, namely we can claim that

$$G_{a,r}(x)=O(x^{a/2+1/4}), \qquad G_{1/2,r}(x)=O(x^{1/2}\log x) \tag{4}$$

in the case specified above, while in case $0\leqq a<\frac{1}{2}$ and $r\geqq 2$ it holds that

$$G_{a,r}(x)=O(x^{(4a+3)/10}). \tag{5}$$

In this note we shall give further developments in the investigation of the conjecture (2) in case $a<\frac{1}{2}$ and $r=2$, namely, we shall state a series representation for $G_{a,2}(x)$ similar to that for $G_{0,2}(x)$ obtained by Wigert [9], an average result for $-\frac{1}{2}<a<\frac{1}{2}$ analogous to that proved by Hardy [5] regarding Dirichlet's divisor problem, and finally

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an Ω-result which follows from Berndt's theorem [1]. The detailed proofs of the following theorems will be given elsewhere.

Theorem 1. *We have for* $-\frac{1}{2}<k<\frac{3}{2}$,

(6) $$G_{k,2}(x)=-\frac{x^{k/2+1/4}}{2^{1/2}\pi^2}f_{1-k}(4\pi\sqrt{x})-x^{k-1}G_{2-k,2}(x)+O(x^{k/2}),$$

so that

(7) $$G_{k,2}(x)=-\frac{x^{k/2+1/4}}{2^{1/2}\pi^2}f_{1-k}(4\pi\sqrt{x})+O(x^{k/2+1/4}),$$

where

(8) $$f_k(x)=\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{5/4+k/2}}\sin\left(\sqrt{n}\,x-\frac{\pi}{4}\right),$$

$\sigma_k(n)$ *being the sum of k-th powers of divisors of n.*

Theorem 2. *We have for every positive* ε

(9) $$\int_1^x \{G_{k,2}(t)\}^2\,dt=O(x^{3/2+k+\varepsilon}),$$

provided that $-\frac{1}{2}<k<\frac{1}{2}$.

Theorem 3. *For every positive* ε *it holds that*

(10) $$x^{-1}\int_1^x |G_{k,2}(t)|\,dt=O(x^{1/4+k/2+\varepsilon}),$$

i.e. Chowla and Walum's conjecture (2) *is true on average if* $-\frac{1}{2}<k<\frac{1}{2}$; *in particular*

(11) $$x^{-1}\int_1^x |G_{0,2}(t)|\,dt=O(x^{1/4+\varepsilon}),$$

i.e. Chowla's conjecture (3) *is true on average.*

Theorem 4. *If* $-\frac{1}{2}<k<\frac{1}{2}$, *then we have*

(12) $$G_{k,2}(x)=\Omega_+(x^{k/2+1/4}(\log x)^{1/4-k/2})$$

and

(13) $$\liminf_{x\to\infty}\frac{G_{k,2}(x)}{x^{k/2+1/4}}=-\infty.$$

Corollary. *If* $R(x,r)$ *denotes the non-trivial error term in the asymptotic formula for*

(14) $$\sum_{n\leq x}(x^r-n^r)\sigma_{-r}(n),$$

then

(15) $$R(x,r)=\Omega_-(x^{(2r-1)/4}(\log x)^{3/4-r/2})$$

and

(16) $$\limsup_{x\to\infty}\frac{R(x,r)}{x^{(2r-1)/4}}=+\infty$$

for $\frac{1}{2}<r<\frac{3}{2}$.

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