38. *On Inclusion Relations between Two Methods of Summability*

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1. Introduction. Let $A=(a_{mn})$ be an infinite matrix. For a given sequence $\{s_n\}$, we set

$$\sigma_m=\sum_{n=0}^{\infty} a_{mn}s_n,$$

which is called the (A) mean of the sequence $\{s_n\}$. If the sequence $\{\sigma_m\}$ is convergent to s, then the sequence $\{s_n\}$ is said to be summable (A) to sum s. If any convergent sequence is necessarily summable (A), then the method of summability (A) is said to be *conservative*. If the sequence $\{\sigma_m\}$ is absolutely convergent, that is,

$$\sum_{m=0}^{\infty} |\sigma_m-\sigma_{m+1}|<+\infty,$$

then the sequence $\{s_n\}$ is said to be absolutely summable (A), or shortly summable $|A|$. If any absolutely convergent sequence is necessarily summable $|A|$, then the method of summability (A) is said to be *absolutely conservative*.

The purpose of this note is to solve the following problems.

(I) If the method of summability (A) is conservative, then is the method (A) absolutely conservative?

(II) If the method of summability (A) is absolutely conservative, then is the method (A) conservative?

In § 2, we show that these problems are negatively solved. In § 3, we prove some theorem concerning the problem (I). When (A) and (B) are methods of summability, we say that the method (B) includes the method (A) and use the notation $(A)\subseteq(B)$, if any sequence summable (A) is necessarily summable (B). We shall now consider the following problems, in which the method (A) is not the unit matrix method, analogous to the above problems (I) and (II).

(I′) If $(A)\subseteq(B)$, then is it true that $|A|\subseteq|B|$?

(II′) If $|A|\subseteq|B|$, then is it true that $(A)\subseteq(B)$?

In § 4, we show that these problems are also negatively solved.

2. Concerning the problems (I) and (II), we state the following theorems.

Theorem 1. *There exists a method of summability (A) such that the method (A) is conservative but not absolutely conservative.*

Theorem 2. *There exists a method of summability* (A) *such that the method* (A) *is absolutely conservative but not conservative.*

For the proof of these theorems we need the following theorems.

Theorem A (Kojima-Schur [2, Theorem 1]). *In order that the method of summability* (A) *should be conservative, it is necessary and sufficient that*

(2.1) *there exists a constant* H *such that*

$$\sum_{n=0}^{\infty} |a_{mn}| < H \qquad \text{for all } m,$$

(2.2) $$a_{mn} \to \delta_n \ (m \to \infty) \qquad \text{for each } n$$

and

(2.3) $$a_m = \sum_{n=0}^{\infty} a_{mn} \to \delta \qquad (m \to \infty).$$

Theorem B (Mears-Sunouchi [4], [5]). *In order that the method of summability* (A) *should be absolutely conservative, it is necessary and sufficient that*

(2.4) *the series* $\sum_{n=0}^{\infty} a_{mn}$ *converges for each* m

and there exists a constant H *such that*

(2.5) $$\sum_{m=0}^{\infty} \left| \sum_{n=k}^{\infty} (a_{m+1,n} - a_{mn}) \right| < H \qquad \text{for } k=0,1,2,\cdots$$

or

(2.5′) $$\sum_{m=0}^{\infty} \left| \sum_{n=0}^{k} (a_{m+1,n} - a_{mn}) \right| < H \qquad \text{for } k=0,1,2,\cdots.$$

Remark. *It is easily seen that* (2.5) *and* (2.5′) *are equivalent.*

We shall now prove Theorems 1 and 2.

Proof of Theorem 1. Let $A=(a_{mn})$ where

$$a_{mn} = (-1)^m (n+1)^{-2} (m+1)^{-1/2} \qquad (m, n = 0, 1, 2, \cdots).$$

Then, for all m,

$$\sum_{n=0}^{\infty} |a_{mn}| \leqq \sum_{n=0}^{\infty} (n+1)^{-2} = \pi^2/6$$

and, for each n,

$$a_{mn} \to 0 \qquad (m \to \infty).$$

Furthermore

$$a_m = \sum_{n=0}^{\infty} a_{mn} = (-1)^m (m+1)^{-1/2} \sum_{n=0}^{\infty} (n+1)^{-2} \to 0 \qquad (m \to \infty).$$

Thus, by Theorem A, the method (A) is conservative. Now we have, for any fixed $k=0,1,2,\cdots$,

$$\sum_{m=0}^{\infty} \left| \sum_{n=k}^{\infty} (a_{m+1,n} - a_{mn}) \right| = \sum_{m=0}^{\infty} \sum_{n=k}^{\infty} (n+1)^{-2} \{(m+2)^{-1/2} + (m+1)^{-1/2}\}$$
$$\geqq (k+1)^{-2} \sum_{m=0}^{\infty} (m+1)^{-1/2} = +\infty.$$

Hence, by Theorem B, the method (A) is not absolutely conservative.

Proof of Theorem 2. Let $A=(a_{mn})$ where

$$a_{mn}=(-1)^n(n+1)^{-1}(m+1)^{-1} \qquad (m, n=0, 1, 2, \cdots).$$

Then it is obvious that (a_{mn}) satisfies the condition (2.4). Furthermore it is easily seen that (a_{mn}) satisfies the condition (2.5). In fact, for any fixed $k=0, 1, 2, \cdots$,

$$\begin{aligned}\sum_{m=0}^{\infty}\left|\sum_{n=k}^{\infty}(a_{m+1,n}-a_{mn})\right| &=\sum_{m=0}^{\infty}\left|\sum_{n=k}^{\infty}(-1)^n(n+1)^{-1}\right|\{(m+2)^{-1}-(m+1)^{-1}\}\\ &=\left|\sum_{n=k}^{\infty}(-1)^n(n+1)^{-1}\right|\sum_{m=0}^{\infty}\{(m+1)^{-1}-(m+2)^{-1}\}\\ &=\left|\sum_{n=k}^{\infty}(-1)^n(n+1)^{-1}\right|\leqq 1,\end{aligned}$$

since the series $\sum_{n=0}^{\infty}(-1)^n(n+1)^{-1}$ converges. Thus, by Theorem B, the method (A) is absolutely conservative. On the other hand, we have

$$\sum_{n=0}^{\infty}|a_{mn}|=\sum_{n=0}^{m}|a_{mn}|=(m+1)^{-1}\sum_{n=0}^{\infty}(n+1)^{-1}=+\infty.$$

Hence, by Theorem A, the method (A) is not conservative.

3. Concerning the problem (I) in § 1, we have the following

Theorem 3. *Let $A=(a_{mn})$, where $a_{mn}\geqq 0$ and, for any fixed n, the sequence $\{a_{mn}\}$ is monotonic. Then, if the method of summability (A) is conservative, it is also absolutely conservative.*

Proof. Let the method (A) be conservative. Then we first remark that, for each m, the series $\sum_{n=0}^{\infty}a_{mn}$ is convergent by (2.1). Now, by Theorem A, there exists a constant H such that

$$\sum_{n=0}^{\infty}|a_{mn}|<H \qquad \text{for all } m.$$

Furthermore, by (2.1) and (2.2), putting $\delta_n=\lim_{m\to\infty}a_{mn}$, it follows that the series $\sum_{n=0}^{\infty}\delta_n$ is convergent. Hence, for all k,

$$\begin{aligned}\sum_{m=0}^{\infty}\left|\sum_{n=0}^{k}(a_{m+1,n}-a_{mn})\right|=\left|\sum_{m=0}^{\infty}\sum_{n=0}^{k}(a_{m+1,n}-a_{mn})\right| &=\left|\lim_{p\to\infty}\sum_{n=0}^{k}\sum_{m=0}^{p}(a_{m+1,n}-a_{mn})\right|\\ &=\left|\lim_{p\to\infty}\sum_{n=0}^{k}(a_{p+1,n}-a_{0n})\right|\\ &=\left|\sum_{n=0}^{k}(\delta_n-a_{0n})\right|\leqq\left|\sum_{n=0}^{\infty}\delta_n-\sum_{n=0}^{\infty}a_{0n}\right|\leqq H.\end{aligned}$$

Thus (a_{mn}) satisfies the condition (2.5′). Therefore, by Theorem B, the method (A) is absolutely conservative.

4. Concerning the problems (I′) and (II′) in § 1, we state the following theorems.

Theorem 4. *There exist two methods of summability* (A) *and* (B) *such that the relation* $(A)\subseteqq(B)$ *holds but the relation* $|A|\subseteqq|B|$ *does not hold.*

Theorem 5. *There exist two methods of summability* (A) *and* (B) *such that the relation* $|A|\subseteqq|B|$ *holds but the relation* $(A)\subseteqq(B)$ *does not hold.*

Proof of Theorem 4. If we put $(A)=(Y)$ and $(B)=(C,1)$ in which (Y) means and $(C,1)$ means of the sequence $\{s_n\}$ are defind by

$$\sigma_0=s_0,\quad \sigma_n=\frac{1}{2}(s_{n-1}+s_n)\quad (n=1,2,3,\cdots)$$

and

$$\tau_n=(s_0+s_1+\cdots+s_n)/(n+1)\qquad (n=0,1,2,\cdots),$$

respectively, then we may prove that (i) the relation $(Y)\subseteqq(C,1)$ holds but (ii) the relation $|Y|\subseteqq|C,1|$ does not hold. The proposition (i) is easily proved by means of the theorem of Riesz (Hardy [2, Theorem 19]). For the proof of the proposition (ii), we use the sequence $\{s_n\}$ such that

$$s_{2n}=0\quad\text{and}\quad s_{2n+1}=1\quad (n=0,1,2,\cdots).$$

Then we have

$$\sigma_n=\frac{1}{2}\qquad (n>1),\qquad \sum_{n=0}^{\infty}|\sigma_n-\sigma_{n+1}|=\frac{1}{2}$$

$$\tau_n=\frac{1}{2}\left(1-\frac{1}{n+1}\right)\quad (n:\text{ even}),\qquad =\frac{1}{2}\quad (n:\text{ odd})$$

and

$$\sum_{n=0}^{\infty}|\tau_n-\tau_{n+1}|\geqq\sum_{n=0}^{\infty}|\tau_{2n}-\tau_{2n+1}|=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2n+1}=+\infty.$$

These facts thus prove the proposition (ii).

Remark. The proposition (ii) was proved by Usha Kakkar [3] using another example.

Proof of Theorem 5. Before going to the proof, we state some definitions and theorems. Given two sequences $\{p_n\}$ and $\{\alpha_n\}$, let us put

$$t_n=(p*\alpha)_n^{-1}\sum_{k=0}^{n}p_{n-k}\alpha_k s_k,$$

where $(p*\alpha)_n=p_0\alpha_n+p_1\alpha_{n-1}+\cdots+p_n\alpha_0\neq 0$. Then t_n is called (N,p,α) mean of the sequence $\{s_n\}$ and defines the method of summability (N,p,α). Especially we use $(\overline{N},\alpha)$ instead of $(N,1,\alpha)$. For these methods of summability, the following theorems are known.

Theorem C. *Let*

(4.1) $p_n>0,\ p_{n+1}/p_n\leqq p_{n+2}/p_{n+1}\leqq 1$ *and* $\alpha_n>0$ $(n=0,1,2,\cdots)$.

Then, in order that the relation $(N,p,\alpha)\subseteqq(\overline{N},\alpha)$ *holds, it is necessary and sufficient that*

$$\sum_{n=0}^{\infty}p_n=+\infty\quad\text{or}\quad\sum_{n=0}^{\infty}\alpha_n=+\infty.$$

Theorem D. *Let the sequences* $\{p_n\}$ *and* $\{\alpha_n\}$ *satisfy the condition* (4.1). *Then the relation* $|N, p, \alpha| \subseteqq |\overline{N}, \alpha|$ *holds.*

These are due to Das [1]. Concerning Theorem C, it is remarked that Das has proved the sufficiency of the condition, but we may easily prove the necessity of the condition from the proof of Das [1] using Theorem A in § 2. We shall now proceed the proof of Theorem 5. Let

$$p_n = \alpha_n = (n+1)^{-2} \qquad (n = 0, 1, 2, \cdots).$$

Then, by Theorems C and D, we may easily prove that the relation $(N, p, \alpha) \subseteqq (\overline{N}, \alpha)$ does not hold and that the relation $|N, p, \alpha| \subseteqq |\overline{N}, \alpha|$ holds. Thus the proof of Theorem is completed.

References

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