37. Multi-Dimensional Generalizations of the Chebyshev Polynomials. II*)

By K. B. DUNN and R. LIDL
Mathematics Department, University of Tasmania, Australia

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4. Proofs. Proof of Lemma 3.1. We have

$$\frac{1}{k}P^{-1/2}_{0,m}=\frac{1}{k}P^{-1/2}_{-m,0}=(u_1^{-1})^m+\cdots+(u_{k+1}^{-1})^m.$$

Thus it is necessary to show that

$$r(z)=z^{k+1}-(b^{-1}x_k)z^k+\cdots+(-1)^k(b^{-1}x_1)z+(-1)^{k+1}b^{-1}$$

has roots $u_1^{-1},\cdots,u_{k+1}^{-1}$. This follows from

$$\prod_{i=1}^{k+1}(z-u_i^{-1})=\sum_{i=0}^{k+1}(-1)^i x_i z^{k+1-i}$$

and $\sigma_i(u_1^{-1},\cdots,u_{k+1}^{-1})=b^{-1}\sigma_{k+1-i}(u_1,\cdots,u_{k+1})$. The proof for $P^{1/2}_{-m,0}$ is similar.

Proof of Lemma 3.2. If we allow $b=0$ in the definition of $P^{-1/2}_{m,0}(\underline{x}\,;b)$, then $u_1,\cdots,u_{k+1}$ are the roots of

$$z^{k+1}-x_1z^k+\cdots+(-1)^k x_k z=z(z^k+\cdots+(-1)^k x_k),$$

where $x_i=\sigma_i(\underline{u})$, $\underline{u}=(u_1,\cdots,u_{k+1})$. One of these roots, say u_{k+1}, is zero. Then

$$\frac{1}{k}P^{-1/2}_{m,0}(\underline{x}\,;0)=u_1^m+\cdots+u_k^m$$

where $u_1,\cdots,u_k$ are the roots of

$$z^k-x_1z^{k-1}+\cdots+(-1)^k x_k.$$

This proves the first formula. The proof for $P^{1/2}_{m,0}(\underline{x}\,;0)$ is analogous.

The proof of Lemma 3.4 follows from Definition 2.1 using methods similar to those in [11].

Proof of Theorem 3.5.

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}&P^{-1/2}_{m,n}s^m t^n\\ &=\sum_m\sum_n\frac{1}{k^2}P^{-1/2}_{m,0}P^{-1/2}_{-n,0}-\frac{1}{k}P^{-1/2}_{m-n,0}s^m t^n\\ &=\left(\frac{1}{k}\sum_m P^{-1/2}_{m,0}s^m\right)\left(\frac{1}{k}\sum_n P^{-1/2}_{-n,0}t^n\right)-\frac{1}{k}\sum_m\sum_n P^{-1/2}_{m-n,0}s^m t^n\\ &=\frac{N_+}{D_+}\frac{N_-}{D_-}-\frac{1}{k}\sum_m\sum_n P^{-1/2}_{m-n,0}s^m t^n.\end{aligned}$$

The last term here can be expressed in terms of D_+, D_-, N_+, N_- by

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using $\frac{1}{k}\sum_m p_{m,0}^{-1/2}s^m=\frac{N_+}{D_+}$, and $P_{0,0}^{-1/2}=k(k+1)$. We obtain

$$\frac{1}{k}\sum_m\sum_n P_{m-n,0}^{-1/2}s^mt^n=\frac{N_+}{D_+(1-st)}+\frac{N_-}{D_-(1-st)}-\frac{k+1}{1-st}$$
$$=\frac{D_+N_-+D_-N_+-(k+1)D_+D_-}{D_+D_-(1-st)}.$$

Thus we have the required result, provided $1-st$ is a factor of $M'=D_+N_-+D_-N_+-(k+1)D_+D_-$. To show that this is the case we firstly consider the term of M' which contains no x_i's, namely the term

$$(k+1)b[1+b(-s)^{k+1}]+(k+1)[b+(-t)^{k+1}]$$
$$-(k+1)[1+b(-s)^{k+1}][b+(-t)^{k+1}]=(k+1)b[1-(st)^{k+1}].$$

Clearly this term is divisible by $1-st$. Secondly we consider the coefficient of x_i, $1\leq i\leq k$, in M'. After some simplification this can be given in the form

$$i(-t)^{k+1-i}[1-(st)^i]+b(k+1-i)(-s)^i[1-(st)^{k+1-i}].$$

Again, $1-st$ is a factor of this term. Finally we consider the remaining terms of M', containing x_ix_j, $i\leq j$. The coefficient of x_ix_j in M' can be expressed in the form

$$(j-i)(-s)^i(-t)^{k+1-j}[1-(st)^{j-i}]$$

and therefore is also divisible by $1-st$. Hence we have shown that $M'=(1-st)M$, for a suitable polynomial M.

The proofs of the recurrence relations given in Theorem 3.6 are elementary and are therefore omitted.

For the proof of Lemma 3.7 and Theorem 3.8 we need some additional properties. Let $\underline{u}=(u_1,\cdots,u_{k+1})$, $\underline{u}^{(j)}=(u_1,\cdots,u_{j-1},u_{j+1},\cdots,u_{k+1})$ and $\mathfrak{U}_{0,0}^{(j)}$ the matrix defined by (2.10) but without the j-th column and the 1st row, i.e. $\mathfrak{U}_{0,0}^{(j)}$ is a $k\times k$ matrix. σ_i denotes the i-th elementary symmetric function. The first auxiliary result is,

Lemma 4.1. $\sum_{j=1}^{k+1}[(-1)^{j-1}u_j^p\sigma_{k-1}(\underline{u}^{(j)})\det\mathfrak{U}_{0,0}^{(j)}]=0,$

where $0\leq i\leq k,\qquad i<p\leq k.$

Proof. Cleary, we have

$$\det\mathfrak{U}_{m,0}=0\qquad\text{for } -k\leq m<0 \tag{4.1}$$

and

$$u_j\sigma_i(\underline{u}^{(j)})=\sigma_{i+1}(\underline{u})-\sigma_{i+1}(\underline{u}^{(j)}). \tag{4.2}$$

Then the proof of the Lemma is a simple induction proof on i.

Proof of Lemma 3.7. Since $P_{m,0}^{1/2}=(\det\mathfrak{U}_{m,0})(\det\mathfrak{U}_{0,0})^{-1}$, we have

$$\frac{1}{k}\sum_{m=0}^{\infty}P_{m,0}^{1/2}s^m=(\det\mathfrak{U}_{0,0})^{-1}\cdot(D_+)^{-1}N',$$

where

$$N'=\sum_{j=0}^{k+1}(-1)^{j-1}u_j^k\det\mathfrak{U}_{0,0}^{(j)}\prod_{\substack{i=1\\i\neq j}}^{k+1}(1-u_is) \tag{4.3}$$

$$=\sum_{j=0}^{k+1}(-1)^{j-1}u_j^k \det \mathfrak{U}_{0,0}^{(j)}\sigma_{k-i}(\underline{u}^{(j)})$$
$$=\begin{cases}0 & \text{if } 0\le i<k \text{ (by Lemma 4.1)}\\ \det \mathfrak{U}_{0,0} & \text{if } i=k.\end{cases}$$

This proves the first formula. The second one follows immediately from the first and Lemma 3.1.

The second auxiliary result is,

Lemma 4.2.

$$\sum_{j=1}^{k+1}(-1)^{j-1}\det \mathfrak{U}_{0,0}^{(j)}u_j^k\prod_{\substack{p=1\\p\ne j}}^{k+1}(1-u_ps)=\det \mathfrak{U}_{0,0} \tag{4.4}$$

$$\sum_{j=1}^{k+1}(-1)^{j+k}\det \mathfrak{U}_{0,0}^{(j)}u_j^{-1}\prod_{\substack{p=1\\p\ne j}}^{k+1}(1-u_p^{-1}s)=b^{-1}\det \mathfrak{U}_{0,0} \tag{4.5}$$

$$\sum_{j=1}^{k+1}(-1)^{j+k}\det \mathfrak{U}_{0,0}^{(j)}\prod_{\substack{p=1\\p\ne j}}^{k+1}(1-u_p^{-1}s)=b^{-1}s\det \mathfrak{U}_{0,0}. \tag{4.6}$$

Proof. The coefficient of $(-s)^p$ in (4.4) is

$$\sum_{j=1}^{k+1}(-1)^{j-1}\det \mathfrak{U}_{0,0}^{(j)}u_j^k\sigma_p(\underline{u}^{(j)})=\begin{cases}0 & \text{if } 0<p\le k\\ \det U_{0,0} & \text{if } p=0;\end{cases}$$

this follows from (4.3) in the proof of Lemma 3.7. Thus (4.4) is correct. Now let $\mathfrak{U}_{0,0}^{(-1,j)}$ denote the matrix obtained from $\mathfrak{U}_{0,0}$ by deleting the first row and j-th column and all entries u_i replaced by u_i^{-1}. Then

$$\sum_{j=1}^{k+1}(-1)^{j-1}\det \mathfrak{U}_{0,0}^{(-1,j)}u_j^{-k}\prod_{\substack{p=1\\p\ne j}}^{k+1}(1-u_p^{-1}s)\det \mathfrak{U}_{0,0}^{(-1)}. \tag{4.7}$$

It is easily verified that

$$b^k\det \mathfrak{U}_{0,0}^{(-1)}=(-1)^{(1/2)(k+1)(k+2)}\det \mathfrak{U}_{0,0}.$$

Similarly,

$$(bu_j^{-1})^{k-1}\det \mathfrak{U}_{0,0}^{(-1,j)}=(-1)^{(1/2)(k+1)}\det \mathfrak{U}_{0,0}^{(j)}.$$

Thus, multiplying both sides of (4.7) by b^{k-1} gives

$$\sum_{j=1}^{k+1}(-1)^{j-1}(-1)^{(1/2)k(k+1)}\det \mathfrak{U}_{0,0}^{(j)}u_j^{-1}\prod_{\substack{p=1\\p\ne j}}^{k+1}(1-u_p^{-1}s)$$
$$=b^{-1}(-1)^{(1/2)(k+1)(k+2)}\det \mathfrak{U}_{0,0}.$$

Multiplication by $(-1)^{(-1/2)(k+1)(k+2)}$ gives the result (4.5). Finally, consider

$$\sum_{j=1}^{k+1}(-1)^{j-1}\det \mathfrak{U}_{0,0}^{(j)}u_j^{k-1}\prod_{\substack{p=1\\p\ne j}}^{k+1}(1-u_ps).$$

The coefficient of $(-s)^p$ is

$$\sum(-1)^{j-1}\det \mathfrak{U}_{0,0}^{(j)}u_j^{k-1}\sigma_p(\underline{u}^{(j)}).$$

If $1<p\le k$ this coefficient is 0, by Lemma 4.1. If $p=0$, then

$$\sum_{j=0}^{k+1}(-1)^{j-1}\det \mathfrak{U}_{0,0}^{(j)}u_j^{k-1}=0.$$

If $p=1$, then

$$\sum_j(-1)^{j-1}\det \mathfrak{U}_{0,0}^{(j)}u_j^{k-1}\sigma_1(\underline{u}^{(j)})=\sum_j(-1)^{j-1}\det \mathfrak{U}_{0,0}^{(j)}u_j^{k-1}\Bigl(\sum_{\substack{i=1\\i\ne j}}^{k+1}u_i\Bigr)$$

$$=\sum_j (-1)^{j-1} \det \mathfrak{U}_{0,0}^{(j)} u_j^{k-1}\left(\sum_{i=1}^{k+1} u_i - u_j\right)$$
$$= -\det \mathfrak{U}_{0,0},$$

since

$$\sum_j (-1)^{j-1} \det \mathfrak{U}_{0,0}^{(j)} u_j^{k-1} = 0.$$

Thus

$$\sum_{j=1}^{k+1} (-1)^{j-1} \det \mathfrak{U}_{0,0}^{(j)} u_j^{k-1} \prod_{\substack{p=1\\p\neq j}}^{k+1} (1-u_p s) = (-\det \mathfrak{U}_{0,0})(-s).$$

Replacing u_j by u_j^{-1} in this equation and then multiplying by b^{k-1} gives the required result (4.6).

Proof of Theorem 3.8. We expand $\det \mathfrak{U}_{m,n}$ about the top row, then about the bottom row to give

$$\det \mathfrak{U}_{m,n} = \sum_{i=1}^{k+1} (-1)^{i-1} u_i^{k+m} \left(\sum_{\substack{j=1\\j\neq i}}^{k+1} (-1)^{j+k} u_j^{-n} A_{ij}\right)$$

where

$$A_{ij} = \begin{vmatrix} u_1^{k-1} \cdots u_{i-1}^{k-1} & u_{i+1}^{k-1} \cdots u_{j-1}^{k-1} & u_{j+1}^{k-1} \cdots u_{k+1}^{k-1} \\ & \cdots\cdots\cdots\cdots & \\ u_1 \quad \cdots u_{i-1} & u_{i+1} \cdots u_{j-1} & u_{j+1} \cdots u_{k+1} \end{vmatrix}$$
$$= b u_i^{-1} u_j^{-1} \det \mathfrak{U}_{0,0}^{(i,j)}.$$

Thus

$$\det \mathfrak{U}_{m,n} = b \sum_i \sum_j (-1)^{i+j+k-1} u_i^{k+m-1} u_j^{-n-1} \det \mathfrak{U}_{0,0}^{(i,j)}.$$

Hence we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_{m,n}^{1/2} s^m t^n = \sum_m \sum_n (\det \mathfrak{U}_{m,n})(\det \mathfrak{U}_{0,0})^{-1} s^m t^n$$
$$= b(\det \mathfrak{U}_{0,0})^{-1} \sum_i \sum_j (-1)^{i+j+k-1} u_i^{k-1} u_j^{-1} \det \mathfrak{U}_{0,0}^{(i,j)} \left(\sum_m (u_i s)^m\right) \times \left(\sum_n (u_j^{-1} t)^n\right)$$
$$= b(\det \mathfrak{U}_{0,0})^{-1} \sum_i \sum_j \frac{(-1)^{i+j+k-1} u_i^{k-1} u_j^{-1} \det \mathfrak{U}_{0,0}^{(i,j)}}{(1-u_i s)(1-u_j^{-1} t)}$$
$$= b(\det \mathfrak{U}_{0,0})^{-1} D_+^{-1} \sum_j \left[\frac{(-1)^{j+k}(1-u_j s) u_j^{-1}}{(1-u_j^{-1} t)} \sum_i (-1)^{i-1} u_i^{k-1} \times \det \mathfrak{U}_{0,0}^{(i,j)} \prod_{\substack{p=1\\p\neq i,j}}^{k+1} (1-u_p s)\right].$$

If we replace $\underline{u}$ by $\underline{u}^{(j)}$ in (4.4), we obtain

$$= b(\det \mathfrak{U}_{0,0})^{-1} D_+^{-1} \sum_j \frac{(-1)^{j+k}(u_j^{-1} - s)}{(1-u_j^{-1} t)} \det \mathfrak{U}_{0,0}^{(j)}$$
$$= b(\det \mathfrak{U}_{0,0})^{-1} (D_+ D_-)^{-1} \sum_j (-1)^{j+k} (u_j^{-1} - s) \det \mathfrak{U}_{0,0}^{(j)} \sum_{\substack{p=1\\p\neq j}}^{k+1} (1-u_p^{-1} t)$$
$$= b(\det \mathfrak{U}_{0,0})^{-1} (D_+ D_-)^{-1} \left[\sum_j (-1)^{j+k} u_j^{-1} \det \mathfrak{U}_{0,0}^{(j)} \prod_p (1-u_p^{-1} t) - s \sum_j (-1)^{j+k} \det \mathfrak{U}_{0,0}^{(j)} \prod_p (1-u_p^{-1} t)\right].$$

If we use (4.5) and (4.6), this gives

$$=b(\det \mathfrak{U}_{0,0})^{-1}(D_+D_-)^{-1}[b^{-1}\det \mathfrak{U}_{0,0}-sb^{-1}t\det \mathfrak{U}_{0,0}]$$
$$=\frac{1-st}{D_+D_-}.$$

Theorem 3.9 and Corollary 3.10 follow from the generating function for $P^{1/2}_{m,n}$ by equating coefficients of $s^m t^n$ for $m,n\geq 0$.

Proof of Lemma 3.11.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P^{1/2}_{m,n}s^m t^n=\frac{1-st}{D_+D_-}=\frac{1}{D_+}\frac{1}{D_-}-\frac{s}{D_+}\frac{t}{D_-}$$
$$=\Big(\sum_{m=0}^{\infty} P^{1/2}_{m,0}s^m\Big)\Big(\sum_{n=0}^{\infty} P^{1/2}_{-n,0}t^n\Big)-\Big(\sum_{m=1}^{\infty} P^{1/2}_{m-1,0}s^m\Big)\Big(\sum_{n=1}^{\infty} P^{1/2}_{-(n-1),0}t^n\Big)$$
$$=\Big(\sum_{m=0}^{\infty} P^{1/2}_{m,0}s^m\Big)P^{1/2}_{0,0}+\Big(\sum_{n=0}^{\infty} P^{1/2}_{-n,0}t^n\Big)P^{1/2}_{0,0}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(P^{1/2}_{m,0}P^{1/2}_{-n,0}$$
$$-P^{1/2}_{m-1,0}P^{1/2}_{-(n-1),0}s^m t^n).$$

Equating coefficients of $s^m t^n$ gives the result.

Proof of Lemma 3.12.

$$\frac{1}{k}\sum_{m=0}^{\infty} P^{-1/2}_{m,0}s^m=\frac{1}{D_+}\sum_{i=0}^{k}(k+1-i)(-1)^i x_i s^i,\qquad \text{from Lemma 3.4,}$$
$$=\Big(\sum_{m=0}^{\infty} P^{1/2}_{m,0}s^m\Big)\Big(\sum_i (k+1-i)(-1)^i x_i s^i\Big),\quad \text{from Lemma 3.7}$$
$$=\sum_{i=0}^{k}\sum_{m=i}^{\infty}(k+1-i)(-1)^i P^{1/2}_{m-i,0}s^m,\quad \text{by changing the index of summation.}$$

Equating coefficients gives the result.

Proof of Theorem 3.13. If we put $m_j=0$, $j\neq i$, in the generating function for $D^{-1/2}_{m_1,\dots,m_k}$, we obtain by simplifying the right hand side

$$\sum_{m_i=0}^{\infty} D^{-1/2}_{0,\dots,0,m_i,0,\dots,0}(\underline{x})s_i^{m_i}=\frac{2-2x_is_i}{1-2x_is_i+\Big(1-\sum_{\substack{j=1\\ j\neq i}}^{k} x_i^2\Big)s_i^2}-1$$
$$=\sum_{m_i=1}^{\infty}\frac{1}{k}P^{-1/2}_{m_i,0}\Big(2x_i\,;\,1-\sum_{\substack{j=1\\ j\neq i}}^{k} x_i^2\Big)s_i^{m_i}-1.$$

Equating coefficients of $s_i^{m_i}$ gives $D^{-1/2}_{0,\dots,0}(\underline{x})=1$ and $D^{-1/2}_{0,\dots,0,m_i,0,\dots,0}$ as given in the theorem. The second part of the theorem is proved similarly.

5. Outlook. The main results of this paper are; suitable definitions, generating functions and recurrence relations for k-dimensional extensions of Chebyshev polynomials. Some open problems about these polynomials are to find partial differential operators for the polynomials (for cases $b=1$, $K=C$, $k=1$ and 2 these operators are known, see, e.g. Rivlin [13], Koornwinder [8], respectively). Of great interest is, of course, to find weight functions for the polynomials $P^{\pm 1/2}_{m,n}(\underline{x}\,;1)$ in case $K=C$, such that these polynomials are orthogonal on a region of C^k. For $k=2$ and complex conjugate variables x_1 and x_2 such a weight function is given in [8] and the corresponding region is bounded by the three-cusped deltoid (or Steiner's hypocycloid). Nothing is known yet about the zeros of the polynomials over C. There are also several algebraic-number theoretic problems in case $K=GF(q)$. For instance, which of the polynomials $P_{m,n}^{\pm 1/2}(\underline{x}; b)$ are permutation polynomials, i.e. for which $m, n \in Z$ does $P_{m,n}^{\pm 1/2}(\underline{x}; b)=a$ have exactly q^{k-1} solutions in $GF(q)^k$ for each $a \in GF(q)$. For $n=0$ partial results are given in [10], the special case $k=1$ is studied in [9], Chapter 4, §§ 8 and 9.

References

[1] L. E. Dickson: The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group. I. Ann. Math., **11**, 65–120 (1896–1897).

[2] ——: Linear Groups and an Exposition of Galois Field Theory. Dover, New York (1901).

[3] K. B. Dunn and R. Lidl: Generalizations of the classical Chebyshev polynomials in two variables (to appear).

[4] R. Eier and R. Lidl: Tschebyscheffpolynome in einer und zwei Variablen. Abh. Math. Sem. Univ. Hamburg, **41**, 17–27 (1974).

[5] A. Erdélyi, M. Magnus, F. Oberhettinger, and F. G. Tricomi: Higher Transcendental Functions. II, McGraw-Hill, New York (1953).

[6] M. Fried: On a conjecture of Schur. Michigan Math. J., **17**, 41–55 (1970).

[7] R. O. Hays: Multi-dimensional extensions of the Chebyshev polynomials. Math. Comp., **27**, 621–624 (1973).

[8] T. H. Koornwinder: Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators. III, IV. Indag. Math., **36**, 375–381 (1974).

[9] H. Lausch and W. Nöbauer: Algebra of Polynomials. North Holland, Amsterdam (1973).

[10] R. Lidl and C. Wells: Chebyshev polynomials in several variables. J. reine angew. Math., **255**, 104–111 (1972).

[11] R. Lidl: Tschebyscheffpolynome in mehreren Variablen. Ibid., **273**, 178–198 (1975).

[12] P. E. Ricci: I polinomi di Tchebycheff in più variabili. Rend. di Matematica, **11**, 295–327 (1978).

[13] T. J. Rivlin: The Chebyshev Polynomials. John Wiley & Sons, New York (1974).

[14] I. Schur: Gesammelte Abhandlungen. Vol. 3, Springer-Verlag, Berlin (1973).

[15] G. Szegö: Orthogonal Polynomials. A.M.S. Colloquium Publ., vol. 23, Amer. Math. Soc., Providence, R.I. (1967).