30. *Cross Ratios as Moduli of Cubic Surfaces*

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1. It is evident from the Cartan's definition of the simple Lie group E_6 that the set of 27 lines upon the general cubic surface in P_3 can be put into a one to one correspondence with the set L of weights of the 27 dimensional representation of E_6 under which the triplets of intersecting lines correspond exactly to the triplets of weights whose sum is equal to 0. (There are two of the representations of this dimension which we regard as the same since they are transposed by the outer automorphisms of E_6.) Such a correspondence is called an *isomorphism* of both sets. By a *distinguished cubic surface* we mean pair (S, α) of a non-singular cubic surface S and an isomorphism α of L onto the set of lines upon S. Two distinguished surfaces (S, α), (S', α') are called *isomorphic* if there is a biregular morphism β of S to S' such that $\beta_* \circ \alpha = \alpha'$ where β_* is the bijection of the lines on S to the lines on S' induced by β. (The isomorphism β is unique, since it is determined by $\beta_* = \alpha' \circ \alpha^{-1}$.) The purpose of this note is to realize the set M of the isomorphism classes of distinguished cubic surfaces as an algebraic manifold and to obtain one of its natural completions by using the cross ratios that Cayley first considered for the cubic surface [1]. We shall now give the following remark since, throughout this note, the emphasis is put on the natural actions of $W(E_6)$ over various objects: The Weyl group $W(E_6)$, acting (transitively) on L, operates on the set of lines on S for every distinguished cubic surface (S, α) through the isomorphism α. Just in this sence $W(E_6)$ is classically called the automorphism group of the 27 lines upon the general cubic surface. $W(E_6)$ further operates on M if one requires $g \in W(E_6)$ to send the isomorphism class of (S, α) to that of $(S, \alpha g^{-1})$.

2. A projective plane that meets a non-singular cubic surface in the union of three lines is called a *tritangent* of the surface. Two tritangents are called *colinear* if their intersection lies entirely on the surface. Through one line on the surface there pass exactly five tritangents. Given four colinear tritangents of the surface, one can consider the cross ratios associated with them, since the totality of planes through one line in P_3 can naturally be regarded as the projective line. We regard all such cross ratios as the invariants of the cubic surface; more precisely, we regard them as k-valued functions on M, where k is the ground field assumed to be algebraically closed and with $\mathrm{ch}(k)\neq 2, 3, 5$. By putting these functions in a row, we obtain a mapping of M into k^{270} which immediately turns out to be an imbedding. (Cayley showed that there are 270 of the cross ratios above which six and six form 45 systems in each of which the elements are transformed from each to other by the composition with the linear fractions permuting $0, 1, \infty$ among themselves. Note that all cross ratios have values in $k-\{0,1\}=P_1-\{0,1,\infty\}$ where P_1 denotes $P_1(k)=k\cup\{\infty\}$. Note also that $W(E_6)$ naturally permutes the cross ratios among themselves, and that it thus operates on k^{270}. The imbedding is obviously $W(E_6)$-equivariant.)

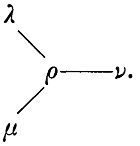
Proposition 1. *As a subset of k^{270}, M is a 4 dimensional closed submanifold.*

We call the closure of M in P_1^{270} $(\supset k^{270})$ the *cross ratio variety* and denote it by C. C might be regarded as the most natural completion of M.

3. To make the structure of C clear enough, we use the following family of cubic surfaces constructed in [3] which is essentially due to Cayley:

$$\begin{aligned}(1)\qquad &\rho W[\lambda X^2+\mu Y^2+\nu Z^2+(\rho-1)^2(\lambda\mu\nu\rho-1)^2W^2+(\mu\nu+1)YZ+(\lambda\nu+1)XZ\\ &+(\lambda\mu+1)XY-(\rho-1)(\lambda\mu\nu\rho-1)W\{(\lambda+1)X+(\mu+1)Y+(\nu+1)Z\}]\\ &+XYZ=0\end{aligned}$$

where X, Y, Z, W are the homogeneous coordinates of P_3, and $(\lambda, \mu, \nu, \rho)$ is a fundamental system of (multiplicative) roots of the simple group D_4 of the adjoint type and with the associated Dynkin diagram:



The multiplicative roots mean exactly the k^*-characters associated with the 1 dimensional eigenspaces of the representation of the maximal torus $T=\{(\lambda, \mu, \nu, \rho)\,;\,\lambda\mu\nu\rho\neq 0\}$ over the Lie algebra of D_4. We should thus consider the family (1) to be defined over T. By modifying the equations in [1], we can write all the tritangents of the cubic surface (1) in the form of linear equations in X, Y, Z, W with coefficients in the function field $k(\lambda, \mu, \nu, \rho)$ of T. Thus, each regular member of the family (1) can be regarded as an element of M, and all the cross ratios associated with (1) are now rational functions on T. It is now just remarkable that they are described in the language of roots as follows: If χ, χ' are roots such that χ/χ' is a root too, then the rational function on T

$$\frac{\chi-1}{\chi'-1}$$

is one of the cross ratios. If $\chi_1, \chi_2 : \chi'_1, \chi'_2$ are roots such that all χ_i/χ'_j are roots and $\chi_1\chi_2 = \chi'_1\chi'_2$, then the rational function

$$\frac{(\chi_1-1)(\chi_2-1)}{(\chi'_1-1)(\chi'_2-1)}$$

is one of the cross ratios. Finally, if $\chi_1, \chi_2, \chi_3, \chi_4 ; \chi'_1, \chi'_2, \chi'_3, \chi'_4$ are roots such that χ_i/χ'_j is a root whenever $i \neq j$ and $\chi_i\chi_j = \chi'_k\chi'_l$ if $\{i, j, k, l\} = \{1, 2, 3, 4\}$, then

$$\frac{(\chi_1-1)(\chi_2-1)(\chi_3-1)(\chi_4-1)}{(\chi'_1-1)(\chi'_2-1)(\chi'_3-1)(\chi'_4-1)}$$

is one of the cross ratios too. These three types exhaust all of the Cayley's cross ratios.

Now, by using these 270 rational functions which actually give rise to a birational mapping of T to C, we can explicitly construct an iterated blowing up, with non-singular centers, of the T-equivariant completion of T associated with the polyhedrical decomposition by the Weyl chambers (cf. [2]), by which the mapping is desingularized, more precisely, becomes a blowing up of C with non-singular center.

Proposition 2. *C is a 4 dimensional algebraic manifold. The boundary $C-M$ is a divisor with normal crossings. There are 76 of its irreducible components which divide themselves into two orbits under the action of $W(E_6)$ over them, one consisting of 36 and the other of 40 components.*

Proposition 2′. *The 36 components in Proposition 2 are the fixed point sets of the 36 reflections of the Weyl group $W(E_6)$. The 40 components are disjoint from each other and can all be blown down simultaneously to the 40 singular points of the space downstair, each of which is locally described as the vertex of the cone over the Veronese imbedding of $P_1 \times P_1 \times P_1$ into P_7.*

We denote the blowing down of C in the proposition by $\check{C}$.

Proposition 3. *There is a family of cubic surfaces over the base $\check{C}$ extending (the restriction to a suitable open subset of* $\mathrm{Spec}(k[\lambda, \mu, \nu, \rho])$ *of) the family* (1) *such that the normal singular members, lying over the images of the 36 components, are exactly the cubic surfaces with only conical singular points, and that the members over the 40 singular points are all isomorphic to $XYZ=0$.*

We finally remark the followings: If some of the 36 components have non-empty intersection, then the reflections corresponding to them commute with each other, and *vice versa*. Each of the 40 components intersects exactly those of the 36 to which the corresponding reflections map it into itself.

The details will be published elsewhere.

References

[1] Cayley, A.: On the triple tangent planes of surfaces of the third order. Collected Papers I, pp. 445–456 (1889).

[2] Kempf, G., Knudsen, F., Mumford, D., and Saint-Donat, B.: Toroidal imbeddings. Lect. Notes in Math., vol. 339, Springer, Berlin-Heidelberg-New York (1973).

[3] Naruki, I., and Sekiguchi, J.: A modification of Cayley's family of cubic surfaces and birational action of $W(E_6)$ over it. Proc. Japan Acad., **56**A, 122–125 (1980).