21. *On the Homogeneous Lüroth Theorem*

By Shigeru IITAKA
Department of Mathematics, University of Tokyo

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§ 1. **Lüroth theorem.** *Let $f, g \in \boldsymbol{C}[X_i, \cdots, X_n]$ such that f is irreducible and suppose that polynomials g and f are algebraically dependent. Then g is a polynomial of f. In particular, if g is also irreducible, then $g=\alpha f+\beta$, α and $\beta \in \boldsymbol{C}$.*

The above statement is equivalent to the Lüroth theorem in the case of polynomials. For the sake of convenience, we begin by giving a proof to the above statement by using logarithmic genera [1].

Proof. Let $A^n = \operatorname{Spec} \boldsymbol{C}[X_1, \cdots, X_n]$, $\Gamma = \operatorname{Spec} \boldsymbol{C}[f, g]$, and $C = \operatorname{Spec} \boldsymbol{C}[f] \rightrightarrows A^1$. Denoting by Γ' the normalization of Γ in A^n, we have the following diagram:

$$\begin{array}{ccc} A^n & \longrightarrow & C \\ \downarrow & \circlearrowright & \uparrow \\ \Gamma' & \longrightarrow & \Gamma \end{array}$$

Diagram 1

Hence $\bar{q}(\Gamma') \leq \bar{q}(A^n)=0$. Since Γ' is normal, we have $\Gamma' \rightrightarrows A^1$ by [3, Example 1]. This implies that $\Gamma' = \operatorname{Spec} \boldsymbol{C}[\theta]$, $\theta \in \boldsymbol{C}[X_1, \cdots, X_n]$. From the inclusions $\boldsymbol{C}[f] \subset \boldsymbol{C}[f, g] \subset \boldsymbol{C}[\theta]$, we infer readily that f is a poly nomial of θ. However, since f is irreducible, f is a linear form of 1 and θ, hence $\boldsymbol{C}[f, g]=\boldsymbol{C}[f]$. Q.E.D.

§ 2. **Quasi-Albanese maps of complements of $\boldsymbol{P}^n$.** Let $F_0, F_1, \cdots, F_r$ be mutually distinct (up to constant multiple) irreducible polynomials with $d_j = \deg F_j$. Consider a sublattice L of $\boldsymbol{Z}^{1+r}$ defined by

$$L=\{\boldsymbol{a} \in \boldsymbol{Z}^{1+r}\,;\, \langle \boldsymbol{a}, \boldsymbol{d}\rangle=0,\ \boldsymbol{d}=(d_0, \cdots, d_r)\}.$$

Let $(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_r)$ be a $\boldsymbol{Z}$-basis of L. Put

$$\Phi_j = \prod F_l^{m(l)}, \qquad \text{where } \boldsymbol{a}_j=(m(1), \cdots, m(r)).$$

Then we have a morphism

$$\alpha=(\Phi_1, \cdots, \Phi_r)\colon V=\boldsymbol{P}^n - \bigcup V_+(F_j) \longrightarrow \boldsymbol{C}^{*r}.$$

α coinsides with the quasi-Albanese map of V (cf. [2]). Denote by Δ the closed image of V by α. Δ is an affine variety whose coordinate ring $\Gamma(\Delta, \mathcal{O}_\Delta)$ is isomorphic to

$$\boldsymbol{C}[\Phi_1, \cdots, \Phi_r, \Phi_1^{-1}, \cdots, \Phi_r^{-1}].$$

Proposition 1. *Suppose that* $\dim \Delta=1$. *Then*

i) *Δ is non-singular,*

ii) *any general fiber of $\alpha\colon V \to \Delta$ is irreducible.*

Proof. This follows easily from the universality of quasi-Albanese maps (see [2]).

Note that ii) is equivalent to the following

ii)′ *The field extension* $C(P^n)/C(\Delta)$ *is an algebraically closed extension.* Here $C(V)$ indicates the function field of V. More precisely, ii)′ is replaced by the following

ii)″ $C[\Phi_1, \cdots, \Phi_r, \Phi_1^{-1}, \cdots, \Phi_r^{-1}]$ *is integrally closed in* $\Gamma(V, \mathcal{O}_V) = \{\Psi/F_0^{s_0}\cdots F_r^{s_r};\ \Psi$ *homogeneous polynomials such that* $\deg \Psi = \sum s_j d_j$, $s_j \geqq 0\}$.

§3. The homogeneous Lüroth theorem. We assume that $d_0 \leqq d_1 \leqq \cdots \leqq d_r$. Put $V_1 = P^n - V_+(F_0) \cup V_+(F_1)$ and $V_j = P^n - V_+(F_0) \cup V_+(F_j)$ $(j \geqq 2)$. Denote by Δ_1 and Δ_j the closed images of the quasi-Albanese maps of V_1 and V_j, respectively. By the property of quasi-Albanese maps, we have the following commutative diagram:

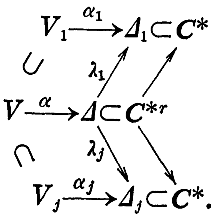

Diagram 2

Then since a general fiber of $\alpha_1: V_1 \to \Delta_1$ is irreducible by Proposition 1, we know that $\lambda_1: \Delta \to \Delta_1$ is birational. Similarly, $\lambda_j: \Delta \to \Delta_j$ is also birational. Hence $C(\Delta_1) = C(\Delta_j)$. Let $\bar{d} = GCD(d_0, d_1)$, and put $\delta_0 = d_1/\bar{d}$ and $\delta_1 = d_0/\bar{d}$. Then $d_0\delta_0 = d_1\delta_1 = LCM(d_0, d_1)$. In place of d_0 and d_1, we consider d_0 and d_j and then we get δ_0' and δ_j'. Hence $d_0\delta_0' = d_j\delta_j' = LCM(d_0, d_j)$. Thus

$$C(\Delta_1) = C(F_1^{\delta_1}/F_0^{\delta_0})$$

and

$$C(\Delta_j) = C(F_j^{\delta_j'}/F_0^{\delta_0'}).$$

From $C(\Delta_1) = C(\Delta_j)$, we derive

$$\frac{F_1^{\delta_1}}{F_0^{\delta_0}} = \frac{\alpha F_j^{\delta_j'} + \beta F_0^{\delta_0'}}{\gamma F_j^{\delta_j'} + \varepsilon F_0^{\delta_0'}},$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \varepsilon \end{pmatrix} \in GL(2, C).$$

We use the following easy lemma.

Lemma 1. *Let* $f, g, h \in C[X_0, X_1, \cdots, X_n]$ *be homogeneous distinct (up to constant multiple) irreducible polynomials such that*

$$\frac{f^a}{g^b} = \frac{\alpha h^c + \beta g^e}{\gamma h^c + \varepsilon g^e},$$

where $a, b, c, e \geqq 1$ and $\alpha\varepsilon - \beta\gamma \neq 0$. Then $\gamma = 0$ and $b = e$.

Proof. From $f^a(\gamma h^c + \varepsilon g^e) = g^b(\alpha h^c + \beta g^e)$, it follows that

$$\gamma h^c + \varepsilon g^e = g^b A, \qquad A \in C[X_0, \cdots, X_n].$$

Hence $\gamma=0$ and $b=e$. Q.E.D.

Thus we have $f^a=\alpha h^c+\beta g^b/\varepsilon$.

Lemma 2. *Let f_1, f_2, f_3 be mutually distinct (up to constant multiple) irreducible polynomials such that $f_1^a+f_2^b+f_3^c=0$ for some $a\geqq b\geqq c\geqq 1$. Then $c=1$.*

This is derived from the classical

Theorem (G. Halphen [1]). *Let Λ be a linear pencil of hypersurfaces on $\boldsymbol{P}^n$. Assume that a general member of Λ is irreducible. Then there are at most two divisors of the form $e_i\Gamma_i$, where $e_i\geqq 2$ and the Γ_i are prime divisors, belonging to Λ.*

The author learned the above theorem from Prof. Jouanolou while he stayed in Japan in 1977.

Accordingly, $\delta_0=\delta_0'$ and $\delta_2'=\cdots=\delta_r'=1$. Therefore,

$$F_j=\alpha_j F_0^{\delta_0}+\beta_j F_j^{\delta_j}$$

for some constants α_j and β_j.

Choosing a suitable $\boldsymbol{Z}$-base of L, we get

$$\Phi_1=F_1^{\delta_1}/F_0^{\delta_0}, \qquad \Phi_j=\alpha_j+\beta_j\Phi_1.$$

Hence, putting $\gamma_j=\alpha_j/\beta_j$, we have

$$\Gamma(\Delta, \mathcal{O}_\Delta)=\boldsymbol{C}[\Phi_1, \Phi_1^{-1}, (\Phi_1+\gamma_2)^{-1}, \cdots, (\Phi_1+\gamma_r)^{-1}].$$

Theorem (the homogeneous Lüroth theorem). *Notations being as in the above, we let G be a homogeneous polynomial whose degree is a multiple ($\geqq 1$) of GCD(d_0, d_1). Choosing $\lambda, \nu\geqq 0$ such that GF_1^λ/F_0^ν has degree 0, we assume that GF_1^λ/F_0^ν is algebraic over $\Gamma(\Delta, \mathcal{O}_\Delta)$. Then*

$$G=c\prod (F_1^{\delta_1}+\Phi_j F_0^{\delta_0})^{e_j}.$$

Proof. Putting $R=\Gamma(V, \mathcal{O}_V)$, $B_1=\Gamma(\Delta, \mathcal{O}_\Delta)[GF_1^\lambda/F_0^\nu]$, and $B=\Gamma(\Delta, \mathcal{O}_\Delta)$, we note the following

Lemma 3. *Let R and B are normal rings finitely generated over a field k and let B_1 be a k-subring of R containing B as a subring. Suppose that* $\dim B=\dim B_1=1$ *and B is integrally closed in R and suppose that* $\bar{q}(\operatorname{Spec} R)=\bar{g}(B)$. *Then $B=B_1$.*

The proof is easy and omitted. Q.E.D.

§ 4. A problem on the classification of surfaces. In this section, we assume $n=2$. Hence $V=\boldsymbol{P}^2-V_+(F_0)\cup\cdots\cup V_+(F_r)$. We shall consider some analogy with Enriques' criterion on irrational ruled surfaces. We assume $\bar{\kappa}(V)=-\infty$ and $\bar{q}(S)=r\geqq 1$. Then the image Δ of the quasi-Albanese map of V turns out to be a curve, since $\bar{\kappa}(\Delta)\geqq 0$. Thus, we arrive at the situation in §§ 2 and 3. Applying Kawamata's theorem [4], we conclude that $C_\lambda=V_+(F_1^{\delta_1}+\lambda F_0^{\delta_0})$ is a rational curve with only one cusp p for almost all λ such that $C^\lambda-\{p\}=A^1$.

Problem. Determine homogeneous irreducible polynomials F_0 and $F_1\in\boldsymbol{C}[X_0, X_1, X_2]$ such that $F_1^{\delta_1}+\lambda F_0^{\delta_0}$ is also irreducible for almost all λ and such that $C_\lambda = V_+(F_1^{\delta_1} + \lambda F_0^{\delta_0})$ is a rational curve with only one cusp p satisfying $C_\lambda - \{p\} = A^1$.

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