

18. Euler's Finite Difference Scheme and Chaos

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1. Introduction. Despite the simple dynamical structure of scalar ordinary differential equations, the corresponding difference equations (Euler's scheme) sometimes exhibit a very complicated dynamical behavior. Such phenomena have recently been understood somewhat systematically from the viewpoint of "chaos" theory, which is in course of powerful development since the work of Li-Yorke [1]. In this short note we shall make clear the importance of the role which asymptotically stable equilibrium points play in the process of chaotic phenomena. In other words, it will be shown that a best-settled equilibrium point in the original differential equation is actually apt to turn into a source of chaos in the corresponding difference equation. Our research here was inspired by R. M. May's example of $\frac{du}{dt} = u(\varepsilon - hu)$.

2. Notation and theorem. Let us consider scalar differential equations of the form

$$(1) \quad \frac{du}{dt} = f(u),$$

where $f(u)$ is continuous in \mathbf{R}^1 . We assume that (1) has at least two equilibrium points one of which is asymptotically stable. As is easily seen, this assumption reduces (after a linear transformation of the unknown if necessary) to the conditions

$$\begin{aligned} (*) & \quad f(0) = f(\bar{u}) = 0 \quad \text{for some } \bar{u} > 0, \\ (*) & \quad f(u) > 0 \quad (0 < u < \bar{u}), \\ (*) & \quad f(u) < 0 \quad (\bar{u} < u < K). \end{aligned}$$

Here the constant K is possibly $+\infty$. Euler's difference scheme for (1) takes the form

$$(2) \quad x_{n+1} = x_n + \Delta t \cdot f(x_n),$$

and henceforth we will adopt the notation $F_{\Delta t}(x) = x + \Delta t f(x)$. Our theorem can now be stated as follows:

Theorem. i) *Let (*) hold. Then there exists a positive constant c_1 such that for any $\Delta t > c_1$ the difference equation (2) is chaotic in the sense of Li-Yorke.*

ii) *Suppose in addition that $K = +\infty$; then there exists another*

constant c_2 , $0 < c_1 < c_2$, such that for any $0 \leq \Delta t \leq c_2$ the map $F_{\Delta t}$ has an invariant finite interval $[0, \alpha_{\Delta t}]$ (i.e., $F_{\Delta t}$ maps $[0, \alpha_{\Delta t}]$ into itself) with $\alpha_{\Delta t} > \bar{u}$. Moreover, when $c_1 < \Delta t \leq c_2$, the above-mentioned chaotic phenomenon occurs in this invariant interval.

Remark. It is not difficult to see that if $f(x)$ is analytic and (1) has no asymptotically stable equilibrium point, then (2) can never be chaotic for any nonnegative value of Δt .

3. Proof of the theorem. For each $\Delta t \geq 0$, let us set

$$M(\Delta t) = \max_{0 \leq x \leq \bar{u}} F_{\Delta t}(x);$$

$$R(\Delta t) = \sup \left\{ x \geq \bar{u} \mid \min_{u \leq z \leq x} F_{\Delta t}(z) \geq 0 \right\};$$

$$r(\Delta t) = \sup \left\{ x \geq \bar{u} \mid \min_{u \leq z \leq x} F_{\Delta t}(z) > 0 \right\}.$$

To be brief, $r(\Delta t)$ is the first positive zero of $F_{\Delta t}(x)$, and $R(\Delta t)$ is the first point where $F_{\Delta t}(x)$ changes sign. As Δt varies, $M(\Delta t)$ ranges over the interval $[\bar{u}, +\infty)$, while $R(\Delta t)$ and $r(\Delta t)$ range over $(\bar{u}, +\infty)$.

Lemma 1. i) $M(s)$ is monotone increasing and continuous in s ;
 ii) $M(0) = \bar{u}$ and $\lim_{s \rightarrow +\infty} M(s) = +\infty$.

Lemma 2. i) $R(s)$ is monotone decreasing and left continuous in s (i.e., $R(s) = R(s-0)$);
 ii) $R(0) = +\infty$ and $\lim_{s \rightarrow +\infty} R(s) = \bar{u}$.

Lemma 3. $r(s) = R(s+0)$. (Hence $r(s-0) = R(s)$ when $s > 0$.)

From this lemma it follows that $r(s)$ is right continuous and that $\bar{u} < r(s) \leq R(s) \leq +\infty$. The strict inequality $r(s) < R(s)$ holds on their discontinuity points. Note that $r(0) = R(0) = +\infty$ so that $R(s)$ is continuous at $s=0$ (in a certain generalized sense; namely, $\lim_{s \rightarrow +0} R(s) = +\infty = R(0)$).

Lemma 4. Suppose $r(\Delta t) \leq M(\Delta t)$. Then there exists a point $x^* \in [0, \bar{u}]$ satisfying

$$(3) \quad 0 < F_{\Delta t}^3(x^*) < x^* < F_{\Delta t}(x^*) < F_{\Delta t}^2(x^*),$$

where F^n denotes the n -th iteration of the map F .

These lemmas are not difficult to verify, and now we are ready for the proof of the theorem. Put

$$c_2 = \sup \{s \geq 0 \mid M(s) - R(s) \leq 0\}.$$

Clearly c_2 is a finite positive number (its positivity follows from the continuity of $M(s) - R(s)$ at $s=0$) and, by the left continuity of $M(s) - R(s)$, we have

$$(4.a) \quad R(s) \geq M(s) \quad (0 \leq s \leq c_2),$$

$$(4.b) \quad R(s) < M(s) \quad (s > c_2).$$

Combining (4.b) and Lemma 3, we get

$$(5) \quad r(s) \leq M(s) \quad (s \geq c_2).$$

So, given any $\Delta t \geq c_2$, there exists a point $x^* \in [0, \bar{u}]$ satisfying the condition (3), which assures (2) to be chaotic in the sense of Li-Yorke (see [1, the last Remark]). Since this condition is a stable property under a small perturbation of $F_{\Delta t}$, we can find a constant c_1 , $0 < c_1 < c_2$, such that (2) is chaotic for any $\Delta t > c_1$. Thus the first statement of the theorem is established.

Suppose now that $K = +\infty$, and let $0 \leq \Delta t \leq c_2$. Then it immediately follows from (4.a) that, given any α with $M(\Delta t) \leq \alpha \leq R(\Delta t)$, the interval $[0, \alpha]$ is invariant under $F_{\Delta t}$. And it is also clear that, when $c_1 < \Delta t \leq c_2$, the restriction of $F_{\Delta t}$ to $[0, \alpha]$ nonetheless possesses a point x^* satisfying (3). So the second statement of the theorem follows. Hence the completion of the proof.

References

- [1] T.-Y. Li and J. A. Yorke: Period three implies chaos. Amer. Math. Monthly, **82**, 985–992 (1975).
- [2] R. M. May: Biological populations with nonoverlapping generation: Stable points, stable cycles, and chaos. Science, **186**, 645–647 (1974).