# 17. Studies on Holonomic Quantum Fields. XII

By Mikio SATO, Tetsuji MIWA, and Michio JIMBO
Research Institute for Mathematical Sciences, Kyoto University

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1979)

In our previous note [1] we have considered a classical scattering problem for 2-dimensional massless Dirac fields, and characterized the "$\tau$-function" $\langle g\otimes g^{-1}\rangle$ of the corresponding Clifford group element. As we shall see in this article, this procedure works in the Minkowski space-time $X^{Min}=\boldsymbol{R}^s$ of an arbitrary dimensionality $s$.

To put the matter somewhat differently, what we do amounts to calculate the following path integrals (or more precisely their product $\tau[A]\tau^*[A]$) in a closed form (see § 1):

$$\tau[A]=\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS_0+iS_{int}}\Big/\int \mathcal{D}\bar{\psi}\mathcal{D}\psi^{iS_0}=\langle \boldsymbol{T}(e^{iS_{int}})\rangle \tag{1}$$

$$\tau^*[A]=\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-iS_0+iS_{int}}\Big/\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-iS_0}=\langle \boldsymbol{T}^*(e^{iS_{int}})\rangle$$

$$S_0=\int d^sx\bar{\psi}(x)(i\partial\!\!\!/-m)\psi(x)$$

$$S_{int}=-\int d^sx\bar{\psi}(x)A\!\!\!/(x)\psi(x).$$

Here $A(x)=(A_\mu(x))$ is a given classical external field. Thus $\log\tau[A]$, when incorporated with the free action, gives the effective action for the "gauge field" $A(x)$. (The integral (1) is formally given by $\det(i\partial\!\!\!/-A\!\!\!/-m)/\det(i\partial\!\!\!/-m)$; however the meaning of an infinite dimensional determinant is obscure and should be made precise.)

Indeed we infer that the time-ordered (resp. anti time-ordered) product $\varphi[A]=\boldsymbol{T}(e^{iS_{int}})$ (resp. $\varphi^*[A]=\boldsymbol{T}^*(e^{iS_{int}})$) is nothing but the element of the Clifford group which induces the rotation $T[A]$ (resp. $T[A]^{-1}$), the classical scattering operator. To see this observe that

$$(i\partial\!\!\!/-A\!\!\!/(x)-m)\boldsymbol{T}(e^{iS_{int}}\psi(x))=0 \tag{2}$$
$$\boldsymbol{T}(e^{iS_{int}}\bar{\psi}(x))(i\overleftarrow{\partial\!\!\!/}+A\!\!\!/(x)+m)=0.$$

An arbitrary matrix element $w(x)=\langle\Phi_1|\boldsymbol{T}(e^{iS_{int}}\psi(x))|\Phi_2\rangle$ or $\overline{w}(x)=\langle\Phi_1|\boldsymbol{T}(e^{iS_{int}}\bar{\psi}(x))|\Phi_2\rangle$ satisfies the same equation (2), respectively. Now in the remote past or future we have

$$\begin{aligned} w(x)\sim w_{in}(x)&=\langle\Phi_1|\varphi[A]\psi(x)|\Phi_2\rangle && (x^0\to-\infty) \\ w_{out}(x)&=\langle\Phi_1|\psi(x)\varphi[A]|\Phi_2\rangle && (x^0\to+\infty) \\ \overline{w}(x)\sim\overline{w}_{in}(x)&=\langle\Phi_1|\varphi[A]\bar{\psi}(x)|\Phi_2\rangle && (x^0\to-\infty) \\ \overline{w}_{out}(x)&=\langle\Phi_1|\bar{\psi}(x)\varphi[A]|\Phi_2\rangle && (x^0\to+\infty). \end{aligned} \tag{3}$$

Along with the definition of $T[A]$, $(\overline{w}_{out}, w_{out})=T[A](\overline{w}_{in}, w_{in})$, (3) shows that $\varphi[A]$ belongs to the Clifford group, and $T[A]=T_{\varphi[A]}$. Similar argument leads to the relation $T[A]=T^{-1}_{\varphi^*[A]}$.

Next we consider the limiting case where the external field $A(x)$ is concentrated on a very thin layer $\Gamma$, so that the transition from the incoming wave to the outgoing one is instantaneous. The rotation $T[A]$ is then a multiplication by a function $M(\xi)$ on this layer. We shall give a variational formula for $\log \tau[T]+\log \tau^*[T]$ as a functional of $M(\xi)$ and $\Gamma$ (see § 2).

We are particularly interested in the case where $M(\xi)$ is a step function. Take $s=2$, $\Gamma=\{\xi=(\xi^0,\xi^1)\in X^{Min}\,|\,\xi^0=a^0\}$ and $M(\xi)=1\ (\xi^1>a^1)$, $=e^{2\pi i l}(\xi^1<a^1)$. In this case the rotation $T$ is nothing but the one induced by $\varphi_F(a\,;l)$ in [2] [3]. The results in [2] [3] are reproduced from our variational formula. A natural generalization of this idea in the higher dimensional case leads one to a non-local field operator of a 2-codimensional extended object (a "bag"), which we shall deal with in subsequent papers.

**1.** Let us prepare some generalities on the orthogonal space of free Dirac spinors with a positive mass $m$. Let $W$ (resp. $\overline{W}$) be the space of wave functions $w={}^t(w_1,\cdots,w_r)$ (resp. $\overline{w}=(\overline{w}_1,\cdots,\overline{w}_r)$) satisfying

(4) $$(i\partial\!\!\!/-m)w(x)=0 \qquad (\text{resp. } \overline{w}(x)(i\overleftarrow{\partial\!\!\!/}+m)=0).$$

Here we have set $\partial\!\!\!/=\sum_{\mu=0}^{s-1}\gamma^\mu\partial_\mu$ with $r\times r$ matrices $\gamma^\mu$ satisfying $[\gamma^\mu,\gamma^\nu]_+ =2(\mu=\nu=0)$, $=-2(\mu=\nu\neq 0)$, $=0(\mu\neq\nu)$, and $\overline{w}(x)i\overleftarrow{\partial\!\!\!/}$ means $i\sum_{\mu=0}^{s-1}\partial_\mu\overline{w}(x)\gamma^\mu$. We define a symmetric inner product in $\tilde{W}=\overline{W}\oplus W=\{\tilde{w}=(\overline{w},w)\,|\,\overline{w}\in\overline{W},\ w\in W\}$ by

(5) $$\langle\tilde{w},\tilde{w}'\rangle=\int_{\text{spacelike}}(\overline{w}(x)d^{s-1}x\cdot w'(x)+\overline{w}'(x)d^{s-1}x\cdot w(x))$$

where $d^{s-1}x=\sum_{\mu=0}^{s-1}\gamma^\mu d^{s-1}x_{\hat{\mu}}$, $d^{s-1}x_{\hat{\mu}}=(-)^\mu dx^0\wedge\cdots\wedge dx^{\mu-1}\wedge dx^{\mu+1}\wedge\cdots\wedge dx^{s-1}$.

We introduce free fields $\psi_\alpha(x)\in\overline{W}$ and $\overline{\psi}_\alpha(x)\in W$ by

(6) $$[\psi_\alpha(x)]_\beta(x')=[\overline{\psi}_\beta(x')]_\alpha(x)=iS(x-x')_{\alpha\beta} \qquad (\alpha,\beta=1,\cdots,r)$$

where $iS(x)=\int\dfrac{d^sp}{(2\pi)^s}e^{-ip\cdot x}\varepsilon(p_0)2\pi\delta(p^2-m^2)(p\!\!\!/+m)$. Then $\tilde{w}$ is expressed as

(7) $$\tilde{w}=\int_{\text{spacelike}}(\overline{w}(x)d^{s-1}x\cdot\psi(x)+\overline{\psi}(x)d^{s-1}x\cdot w(x))$$

where $\psi(x)={}^t(\psi_1(x),\cdots,\psi_r(x))$ and $\overline{\psi}(x)=(\overline{\psi}_1(x),\cdots,\overline{\psi}_r(x))$. The vacuum expectation value reads

(8) $$\langle\psi_\alpha(x)\overline{\psi}_\beta(x')\rangle=iS^{(+)}(x-x')_{\alpha\beta}$$

where $iS^{(\pm)}(x)_{\alpha\beta}=\pm\int\dfrac{d^sp}{(2\pi)^s}e^{-ip\cdot x}\theta(\pm p_0)2\pi\delta(p^2-m^2)(p\!\!\!/+m)$.

Given a linear operator $\tilde{F}$ in $\tilde{W}$ such that $\tilde{F}(\overline{W})\subset\overline{W}$ and $\tilde{F}(W)$

$\subset W$, we define its kernel $(\bar{F}(x,x'), F(x,x'))$ by $\bar{F}(x,x')_{\alpha\beta}=\langle\tilde{F}\psi_\alpha(x), \bar{\psi}_\beta(x')\rangle$ and $F(x,x')_{\alpha\beta}=\langle\psi_\alpha(x), \tilde{F}\bar{\psi}_\beta(x')\rangle$, or equivalently by $\tilde{F}\psi_\alpha(x) =\sum_{\beta=1}^r\int\bar{F}(x,x')_{\alpha\beta}d^{s-1}x'\psi_\beta(x')$ and $\tilde{F}\bar{\psi}_\alpha(x)=\sum_{\beta=1}^r\int\bar{\psi}_\beta(x')d^{s-1}x'F(x',x)_{\beta\alpha}$.
For example we have the following correspondence.

(9) $1\leftrightarrow(iS(x-x'),\ iS(x-x')),\quad E_\pm\leftrightarrow(iS^{(\mp)}(x-x'),\ iS^{(\pm)}(x-x'))$.

Now we shall consider $\tilde{W}\otimes C^l=\{\tilde{w}=(\tilde{w}^{(1)},\cdots,\tilde{w}^{(l)})\,|\,\tilde{w}^{(j)}\in\tilde{W}\ (j=1,\cdots,l)\}$. Let $A(x)=(A_\mu(x))$ be an $s$-tuple of smooth $l\times l$ matrix-valued function, which falls off for $x^0\to\pm\infty$. The classical scattering matrix $T[A]$ for the scattering problem

(10) $\qquad (i\partial\!\!\!/-A\!\!\!/(x)-m)w(x)=0,\qquad \bar{w}(x)(i\overleftarrow{\partial\!\!\!/}+A\!\!\!/(x)+m)=0$

is given by the following kernel.

(11)
$$([iS(1-A\!\!\!/S_{adv})^{-1}(1-A\!\!\!/S_{ret})](x,x'),$$
$$[(1-S_{adv}A\!\!\!/)(1-S_{ret}A\!\!\!/)^{-1}iS](x,x')),$$

where $S_{\substack{ret\\adv}}(x)=\pm\theta(\pm x^0)S(x)$. In (11) $S$, 1, etc. are regarded as integral operators on $X^{Min}$ with kernels $S(x-x')$, $\delta^s(x-x')$, etc. From (9) and (11) the kernels for $E_++E_-T$ and $E_++E_-T^{-1}$ are known to be

(12)
$$([iS(1-A\!\!\!/S_{adv})^{-1}(1-A\!\!\!/S_c)](x,x'),\ [(1-S_cA\!\!\!/)(1-S_{ret}A\!\!\!/)^{-1}iS](x,x')),$$
$$([iS(1-A\!\!\!/S_{ret})^{-1}(1-A\!\!\!/S_c^*)](x,x'),\ [(1-S_c^*A\!\!\!/)(1-S_{adv}A\!\!\!/)^{-1}iS](x,x')),$$

respectively. Then using (6) in [1] we have

(13)
$$\log\tau[A]+\log\tau^*[A]$$
$$=\text{trace}\log(1-S_cA\!\!\!/)+\text{trace}\log(1-S_c^*A\!\!\!/)$$
$$-\text{trace}\log(1-S_{ret}A\!\!\!/)-\text{trace}\log(1-S_{adv}A\!\!\!/),$$

or equivalently

(14)
$$\delta\log\tau[A]+\delta\log\tau^*[A]=-\int d^sx\,\text{trace}\,\delta A\!\!\!/(x)\Psi(x,x;A)$$

where

(15)
$$\Psi(x,x';A)=S_c^A(x,x')+S_c^{*A}(x,x')-S_{ret}^A(x,x')-S_{adv}^A(x,x').$$

The Green's functions $S_c^A$, $S_c^{*A}$, $S_{ret}^A$, $S_{adv}^A$ are characterized in the same way as in [1]. We note that $\Psi(x,x;A)$ is well-defined, although individual terms $S_c^A(x,x)$, $S_c^{*A}(x,x)$, etc. are divergent.

**2.** The $\tau$-functions $\tau[A]$, $\tau^*[A]$ depend on $A$ only through the rotation $T=T[A]$. If we regard them as functionals of $T$ and employ the notation $\tau[T]$, $\tau^*[T]$ (the product $\tau[T]\tau^*[T]$ here corresponds $\tau[T]$ in [1]), the variational formula X-(7) [1] reads

(16)
$$2\delta\log\tau[T]+2\delta\log\tau^*[T]$$
$$=\text{trace}\,\delta T\cdot T^{-1}(-Y_+^{-1}E_+Y_++Z_-^{-1}E_+Z_-).$$

Here the kernel functions for the operators in $\tilde{W}$

(17)
$$\tilde{F}=Y_+^{-1}E_+Y_+=E_+(E_++TE_-)^{-1}=\sum_{n=0}^\infty E_+((1-T)E_-)^n$$
$$\tilde{G}=Z_-^{-1}E_+Z_-=(E_++E_-T^{-1})^{-1}E_+=\sum_{n=0}^\infty(E_-(1-T^{-1}))^nE_+,$$

along with those for $\tilde{F}'=-Y_-^{-1}E_-Y_+$, $\tilde{G}'=-Z_-^{-1}E_-Z_+$, are characterized in terms of $T$ as follows. For fixed $x_0$ we set $F_{x_0}(x)=F(x, x_0)$, $F'_{x_0}(x)=F'(x, x_0)$ (resp. $\bar{F}_{x_0}(x)=\bar{F}(x_0, x)$, $\bar{F}'_{x_0}(x)=\bar{F}'(x_0, x)$). Then these are unique elements of $W$ (resp. $\bar{W}$) satisfying

$$(18)\quad \begin{aligned} &E_-(F_{x_0})=0, \quad E_+(F'_{x_0})=0, \quad F_{x_0}(x)-(TF'_{x_0})(x)=iS(x-x_0)\\ &E_-(\bar{F}_{x_0})=0, \quad E_+(\bar{F}'_{x_0})=0, \quad \bar{F}_{x_0}(x)-(T\bar{F}'_{x_0})(x)=iS(x_0-x).\end{aligned}$$

Likewise $\bar{G}_{x_0}(x)=\bar{G}(x, x_0)$, $\bar{G}'_{x_0}(x)=\bar{G}'(x, x_0)\in W$ (resp. $G_{x_0}(x)=G(x_0, x)$, $G'_{x_0}(x)=G'(x_0, x)\in \bar{W}$) satisfy

$$(19)\quad \begin{aligned} &E_+(\bar{G}_{x_0})=0, \quad E_-(\bar{G}'_{x_0})=0, \quad \bar{G}_{x_0}(x)-(T\bar{G}'_{x_0})(x)=iS(x-x_0)\\ &E_+(G_{x_0})=0, \quad E_-(G'_{x_0})=0, \quad G_{x_0}(x)-(TG'_{x_0})(x)=iS(x_0-x).\end{aligned}$$

Now we consider the limiting case where the external field $A(x)$ is concentrated on a thin layer $\Gamma$, a spacelike hypersurface in the Minkowski space-time $X^{Min}$. The rotation $T=T[A]$ then reduces to the multiplication operator on $\Gamma$

$$(20)\quad \begin{aligned} &T(\psi^{(j)}(\xi))=\sum_{k=1}^{l}(M(\xi)^{-1})_{jk}\psi^{(k)}(\xi)\\ &T(\bar{\psi}^{(j)}(\xi))=\sum_{k=1}^{l}\bar{\psi}^{(k)}(\xi)M(\xi)_{kj}, \qquad \xi\in\Gamma.\end{aligned}$$

Here $M(\xi)$ denotes a smooth matrix-valued function on $\Gamma$, assumed to be close to the unity. The kernel representation of $T$ reads

$$(21)\quad \begin{aligned} &\bar{T}(x, x')=\int_\Gamma iS(x-\xi)M(\xi)^{-1}d^{s-1}\xi\cdot iS(\xi-x')\\ &T(x, x')=\int_\Gamma iS(x-\xi)M(\xi)d^{s-1}\xi\cdot iS(\xi-x').\end{aligned}$$

Setting $\tilde{F}_1=\tilde{F}-E_+$, $\tilde{G}_1=\tilde{G}-E_+$ we have also

$$(22)\quad \begin{aligned} \bar{F}_1(x, x')=\sum_{n=1}^{\infty}\int\cdots\int & iS^{(+)}(x-\xi_1)(1-M(\xi_1)^{-1})d^{s-1}\xi_1 iS^{(+)}(\xi_1-\xi_2)\\ &\times(1-M(\xi_2)^{-1})d^{s-1}\xi_2\cdots iS^{(+)}(\xi_{n-1}-\xi_n)(1-M(\xi_n)^{-1})\\ &\qquad d^{s-1}\xi_n iS^{(-)}(\xi_n-x')\end{aligned}$$

$$\begin{aligned} F_1(x, x')=\sum_{n=1}^{\infty}\int\cdots\int & iS^{(+)}(x-\xi_1)(1-M(\xi_1))d^{s-1}\xi_1 iS^{(-)}(\xi_1-\xi_2)\\ &\times(1-M(\xi_2))d^{s-1}\xi_2\cdots iS^{(-)}(\xi_{n-1}-\xi_n)(1-M(\xi_n))\\ &\qquad d^{s-1}\xi_n iS^{(-)}(\xi_n-x')\end{aligned}$$

$$(23)\quad \begin{aligned} G_1(x, x')=\sum_{n=1}^{\infty}\int\cdots\int & iS^{(-)}(x-\xi_1)(1-M(\xi_1))d^{s-1}\xi_1 iS^{(+)}(\xi_1-\xi_2)\\ &\times(1-M(\xi_2))d^{s-1}\xi_2\cdots iS^{(+)}(\xi_{n-1}-\xi_n)(1-M(\xi_n))\\ &\qquad d^{s-1}\xi_n iS^{(+)}(\xi_n-x')\end{aligned}$$

$$\begin{aligned} G_1(x, x')=\sum_{n=1}^{\infty}\int\cdots\int & iS^{(-)}(x-\xi_1)(1-M(\xi_1)^{-1})d^{s-1}\xi_1 iS^{(-)}(\xi_1-\xi_2)\\ &\times(1-M(\xi_2)^{-1})d^{s-1}\xi_2\cdots iS^{(-)}(\xi_{n-1}-\xi_n)(1-M(\xi_n)^{-1})\\ &\qquad d^{s-1}\xi_n iS^{(+)}(\xi_n-x').\end{aligned}$$

All integrals are to be carried out on $\Gamma$. Notice that these are well defined even when $x=x'\in\Gamma$.

If we vary the matrix $M(\xi)$ keeping $\Gamma$ fixed, the variation of the $\tau$-function is given by

(24)
$$\delta \log \tau[T]+\delta \log \tau^*[T]$$
$$=\int_\Gamma \text{trace } \delta M(\xi)\cdot M(\xi)^{-1}(-F_1(\xi,\xi)+G_1(\xi,\xi))d^{s-1}\xi$$
$$=\int_\Gamma \text{trace } \delta M(\xi)\cdot M(\xi)^{-1}(\overline{F}_1(\xi,\xi)-\overline{G}_1(\xi,\xi))d^{s-1}\xi.$$

Next we vary $\Gamma$ while preserving the matrix $M(\xi)$ in the following sense. Let $\sum_{\mu=0}^{s-1}\rho^\mu(\xi)\partial_\mu$ be a vector field on $\Gamma$. For small $\rho=(\rho^0,\cdots,\rho^{s-1})$ we set $\Gamma^\rho=\{\xi^\rho=\xi+\rho(\xi)\,|\,\xi\in\Gamma\}$ and $M^\rho(\xi^\rho)=M(\xi)$ $(\xi\in\Gamma)$. We denote by $T[\rho]$ the rotation corresponding to $(\Gamma^\rho, M^\rho)$, and by $\delta T$ the variation of $T[\rho]$ at $\rho=0$ as a functional of $\rho$. Then the kernel representation of $\delta T$ is given by

(25)
$$\overline{\delta T}(x,x')=\int_\Gamma \sum_{\mu=0}^{s-1}\delta\rho^\mu(\xi)iS(x-\xi)d^{s-1}\xi\cdot(n_\mu\varkappa\partial\!\!\!/-\partial_\mu)M(\xi)^{-1}\cdot iS(\xi-x')$$
$$\delta T(x,x')=\int_\Gamma \sum_{\mu=0}^{s-1}\delta\rho^\mu(\xi)iS(x-\xi)d^{s-1}\xi\cdot(n_\mu\varkappa\partial\!\!\!/-\partial_\mu)M(\xi)\cdot iS(\xi-x')$$

Here we have set $\varkappa=\sum_{\mu=0}^{s-1}\gamma^\mu n_\mu(\xi)$ with $n(\xi)=(n_0(\xi),n_1(\xi),\cdots,n_{s-1}(\xi))$ denoting the unit normal of $\Gamma$. Notice that $n_\mu\varkappa\partial\!\!\!/-\partial_\mu$ is a tangential vector field relative to $\Gamma$. Accordingly the variational formula (24) remains valid, provided that we replace $\delta M(\xi)$ by $\sum_{\mu=0}^{s-1}\delta\rho^\mu(\xi)\cdot(n_\mu\varkappa\partial\!\!\!/-\partial_\mu)\cdot M(\xi)$.

## References

[1] M. Sato, T. Miwa, and M. Jimbo: Proc. Japan Acad., **54**A, 309–313 (1978).
[2] ——: Ibid., **54**A, 36–41, 221–225 (1978).
[3] ——: RIMS preprint, no. 260 (1978); ibid., no. 263 (1978).