# 16. The Implicit Function Theorem for Ultradifferentiable Mappings

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Let $M_p$, $p=0,1,2,\cdots$, be a sequence of positive numbers. An infinitely differentiable function $f$ on an open set $U$ in $\boldsymbol{R}^n$ is said to be an *ultradifferentiable function of class* $\{M_p\}$ (*resp. of class* $(M_p)$) if for each compact set $K$ in $U$ there are constants $h$ and $C$ (resp. and each $h>0$ there is a constant $C$) such that

$$\sup_{x\in K}|D^\alpha f(x)|\leqq Ch^{|\alpha|}M_{|\alpha|}, \qquad |\alpha|=0,1,2,\cdots.$$

A mapping $F=(f_1,\cdots,f_m)$ from an open set $U$ in $\boldsymbol{R}^n$ into $\boldsymbol{R}^m$ is said to be *ultradifferentiable of class* $\{M_p\}$ (resp. $(M_p)$) if all components $f_i$ are ultradifferentiable functions of class $\{M_p\}$ (resp. $(M_p)$).

We assume that $M_p$ satisfies the following conditions:

$$M_0=M_1=1; \tag{1}$$

There is a constant $H$ such that

$$(M_q/q!)^{1/(q-1)}\leqq H(M_p/p!)^{1/(p-1)}, \qquad 2\leqq q\leqq p; \tag{2}$$

Furthermore in case of class $(M_p)$

$$\frac{M_p}{pM_{p-1}}\to\infty \quad \text{as} \quad p\to\infty. \tag{3}$$

Then we have

**The inverse mapping theorem.** *If $F=(f_1,\cdots,f_n)$ is an ultradifferentiable mapping of class $\{M_p\}$ (resp. $(M_p)$) from an open set $U$ in $\boldsymbol{R}^n$ into an open set $V$ in $\boldsymbol{R}^n$ and if the Jacobian*

$$\frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}=\det\left(\frac{\partial f_i}{\partial x_j}\right)$$

*does not vanish at $x^0$ in $U$, then there exist an open neighborhood $U_0$ of $x^0$ in $U$ and an open neighborhood $V_0$ of $y^0=F(x^0)$ in $V$ such that $F$ restricted to $U_0$ is a homeomorphism onto $V_0$ and the inverse on $V_0$ is an ultradifferentiable mapping of class $\{M_p\}$ (resp. $(M_p)$).*

**Proof.** By the inverse mapping theorem for $C^\infty$ mappings there are open neighborhoods $U_0$ and $V_0$ such that $F: U_0\to V_0$ is a $C^\infty$ diffeomorphism. We may assume that the inverse matrix of $(\partial f_i/\partial x_j)$ is uniformly bounded on $U_0$. To estimate the derivatives of the inverse mapping $F^{-1}=(g_1,\cdots,g_n): V_0\to U_0$, we assume that 0 is an arbitrary point in $U_0$ and $F$ maps it to 0 in $V_0$.

Let $(a_{ij})$ be the inverse matrix of $(\partial f_i/\partial x_j)$ at 0 in $U_0$. We set

$$\varphi_i(x)=x_i-\sum_{j=1}^{n} a_{ij}f_j(x), \qquad i=1, \cdots, n.$$

First we consider the case of ultradifferentiable mapping of class $\{M_p\}$. Then there are constants $h$ and $C$ such that

$$\varphi_i(x)\ll C\sum_{p=2}^{\infty}\frac{M_p}{p\,!}(ht)^p,$$

where

$$t=x_1+\cdots+x_n.$$

This means that the formal Taylor expansion of the left hand side is majorized by the right hand side.

If $U_0$ is relatively compact in $U$, we can choose the same constants $h$ and $C$ independent of the arbitrary point 0 in $U_0$.

Since the components $g_i(y)$ of $F^{-1}$ are the solutions of the system of equations

$$g_i(y)=\sum_{j=1}^{n} a_{ij}y_j+\varphi_i(g_1(y), \cdots, g_n(y)), \qquad i=1, \cdots, n,$$

each $g_i(y)$ is majorized by the formal solution $\psi(s)$ of the equation

$$\psi(s)=Bs+C\sum_{p=2}^{\infty}\frac{M_p}{p\,!}(hn\psi(s))^p,$$

where

$$s=y_1+\cdots+y_n$$

and $B$ is a bound of the absolute values $|a_{ij}|$ on $U_0$.

By the Lagrange expansion theorem the coefficient $b_r$ of

$$\psi(s)=b_1Bs+b_2(Bs)^2+\cdots+b_r(Bs)^r+\cdots$$

is given by

$$b_r=\frac{1}{r\,!}\left[\left(\frac{d}{dt}\right)^{r-1}\left(\frac{t}{k(t)}\right)^r\right]_{t=0},$$

where

$$k(t)=t-C\sum_{p=2}^{\infty}\frac{M_p}{p\,!}(hnt)^p.$$

Hence we have by condition (2)

$$\begin{aligned}
b_r&=\frac{1}{r\,!}\left[\left(\frac{d}{dt}\right)^{r-1}\left\{\sum_{q=0}^{\infty}\left(Chn\sum_{p=1}^{r-1}\frac{M_{p+1}}{(p+1)\,!}(hnt)^p\right)^q\right\}^r\right]_{t=0}\\
&\leqq\frac{1}{r\,!}\left[\left(\frac{d}{dt}\right)^{r-1}\left\{\sum_{q=0}^{\infty}\left(Chn\sum_{p=1}^{\infty}\left(H\left(\frac{M_r}{r\,!}\right)^{1/(r-1)}hnt\right)^p\right)^q\right\}^r\right]_{t=0}\\
&=\frac{1}{r\,!}\left[\left(\frac{d}{dt}\right)^{r-1}\left\{\sum_{q=0}^{\infty}(Chn)^q\sum_{p=0}^{\infty}\binom{p+q-1}{p}\left(H\left(\frac{M_r}{r\,!}\right)^{1/(r-1)}hnt\right)^{p+q}\right\}^r\right]_0\\
&=\frac{1}{r\,!}\left[\left(\frac{d}{dt}\right)^{r-1}\left\{\sum_{p=0}^{\infty}Chn(Chn+1)^{p-1}\left(H\left(\frac{M_r}{r\,!}\right)^{1/(r-1)}hnt\right)^p\right\}^r\right]_0\\
&\leqq\frac{1}{r\,!}\left[\left(\frac{d}{dt}\right)^{r-1}\sum_{p=0}^{\infty}\binom{r+p-1}{p}\left\{(Chn+1)H\left(\frac{M_r}{r\,!}\right)^{1/(r-1)}hnt\right\}^p\right]_0
\end{aligned}$$

$$=\frac{1}{r}\binom{2r-2}{r-1}\{(Chn+1)Hhn\}^{r-1}\frac{M_r}{r!}$$

$$\leqq\{4(Chn+1)Hhn\}^{r-1}\frac{M_r}{r!}.$$

We have therefore

$$g_i(y)\ll\frac{1}{4(Chn+1)Hhn}\sum_{p=1}^{\infty}\frac{M_p}{p!}(4B(Chn+1)Hhn)^p s^p.$$

This shows that

$$(4)\qquad |D^\alpha g_i(0)|\leqq B(4B(Chn+1)Hhn)^{|\alpha|-1}M_{|\alpha|},\ |\alpha|\geqq 1,$$

proving that the inverse mapping $F^{-1}$ is ultradifferentiable of class $\{M_p\}$ on $V_0$.

In case $F$ is ultradifferentiable of class $(M_p)$, the proof is modified as follows. Let $h$ be an arbitrary positive number. Then we can find a constant $C$ and $p_0$ independent of the arbitrary point 0 in $U_0$ such that

$$\varphi_i(x)\ll C\sum_{p=2}^{p_0}\frac{M_p}{p!}(ht)^p+\sum_{p=p_0+1}^{\infty}\frac{M_p}{p!}(ht)^p.$$

It follows from condition (3) that if $r_0$ is sufficiently large, then

$$CM_p/p!\leqq(H(M_r/r!)^{1/(r-1)})^{p-1}$$

for $2\leqq p\leqq p_0$ and $r\geqq r_0$, where $H$ is the constant in condition (2).

Hence the coefficient $b_rB^r$ of formal solution $\psi(s)$ of

$$\psi(s)=Bs+C\sum_{p=2}^{p_0}\frac{M_p}{p!}(hn\psi(s))^p+\sum_{p=p_0+1}^{\infty}\frac{M_p}{p!}(hn\psi(s))^p$$

is estimated for $r\geqq r_0$ as

$$b_r=\frac{1}{r!}\left[\left(\frac{d}{dt}\right)^{r-1}\left\{\sum_{q=0}^{\infty}\left(Chn\sum_{p=1}^{p_0-1}\frac{M_{p+1}}{(p+1)!}(hnt)^p+hn\sum_{p=p_0}^{r-1}\frac{M_{p+1}}{(p+1)!}(hnt)^p\right)^q\right\}^r\right]_0$$

$$\leqq\frac{1}{r!}\left[\left(\frac{d}{dt}\right)^{r-1}\left\{\sum_{q=0}^{\infty}\left(hn\sum_{p=1}^{\infty}\left(H\left(\frac{M_r}{r!}\right)^{1/(r-1)}hnt\right)^p\right)^q\right\}^r\right]_0$$

$$\leqq(4(hn+1)Hhn)^{r-1}\frac{M_r}{r!}.$$

If $r<r_0$, we have (4) for some $C$. Therefore if a $k>0$ is given and we take an $h>0$ so that $4(hn+1)Hhn\leqq k$, then we find that

$$(5)\qquad |D^\alpha g_i(0)|\leqq Ck^{|\alpha|}M_{|\alpha|},\qquad |\alpha|\geqq 1,$$

for a constant $C$ independent of the point 0 in $V_0$.

We note that condition (3) is indispensable in the case of class $(M_p)$, because the theorem does not hold for the class $(p!)$ of entire functions.

Now the following theorem is an easy consequence.

**The implicit function theorem.** *If $F=(f_1(x_1,\cdots,x_n),\cdots,f_m(x_1,\cdots,x_n))$ is an ultradifferentiable mapping of class $\{M_p\}$ (resp. $(M_p)$) from an open neighborhood $U$ of 0 in $\boldsymbol{R}^n$ into $\boldsymbol{R}^m$ with $m\leqq n$ such that $F(0)=0$ and if the Jacobian*

$$\frac{\partial(f_1, \cdots, f_m)}{\partial(x_1, \cdots, x_m)} = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,\cdots,m}$$

*does not vanish at* 0, *then there is a unique diffeomorphism* $G=(g_1(y_1, \cdots, y_n), \cdots, g_n(y_1, \cdots, y_n))$ *of class* $\{M_p\}$ *(resp.* $(M_p)$*) of an open neighborhood* $V_0$ *of* 0 *in* $\boldsymbol{R}^n$ *onto an open neighborhood* $U_0$ *of* 0 *in* $\boldsymbol{R}^n$ *such that*

$$f_i(g_1(y_1, \cdots, y_n), \cdots, g_n(y_1, \cdots, y_n)) = y_i, \qquad i=1, \cdots, m,$$

*and*

$$g_j(y_1, \cdots, y_n) = y_j, \qquad j=m+1, \cdots, n.$$

We have also

**The rank theorem.** *If* $F=(f_1, \cdots, f_m)$ *is an ultradifferentiable mapping of class* $\{M_p\}$ *(resp.* $(M_p)$*) from an open neighborhood* $U$ *of* 0 *in* $\boldsymbol{R}^n$ *into an open neighborhood* $V$ *of* 0 *in* $\boldsymbol{R}^m$ *such that* $F(0)=0$ *and if the differential* $dF: TU \to TV$ *is of constant rank* $r$ *on* $U$, *then there are neighborhoods* $U_0 \subset U$ *and* $U_1$ *of* 0 *in* $\boldsymbol{R}^n$ *and* $V_0 \subset V$ *and* $V_1$ *of* 0 *in* $\boldsymbol{R}^m$ *and diffeomorphisms* $\Phi: U_0 \to U_1$ *and* $\Psi: V_0 \to V_1$ *of class* $\{M_p\}$ *(resp.* $(M_p)$*) such that* $\Phi(0)=0$ *and* $\Psi(0)=0$ *and that* $G=\Psi \circ F \circ \Phi^{-1}: U_1 \to V_1$ *maps* $(x_1, \cdots, x_n)$ *to* $(x_1, \cdots, x_r, 0, \cdots, 0)$.

W. Rudin [1] has shown that when $M_p$ satisfies the Denjoy-Carleman condition of non-quasi-analyticity and the logarithmic convexity, the condition

$$(M_q/q!)^{1/q} \leqq H(M_p/p!)^{1/p}, \qquad 1 \leqq q \leqq p, \tag{6}$$

is equivalent to the property that $1/f$ is ultradifferentiable of class $\{M_p\}$ whenever $f$ is an ultradifferentiable function of class $\{M_p\}$ on $\boldsymbol{R}$ such that $\inf |f(x)| > 0$. Our condition (2) is stronger than but not far from his condition (6).

The Gevrey sequence $p!^s$ clearly satisfies conditions (1) and (2) for $s \geqq 1$ and (3) for $s > 1$. Thus the implicit function theorem holds for Gevrey classes $\{p!^s\}$ for $s \geqq 1$ and $(p!^s)$ for $s > 1$. Since the ultradifferentiable functions of class $\{p!\}$ are exactly the real analytic functions, our theorem includes the implicit function theorem for real analytic mappings.

## Reference

[1] W. Rudin: Division in algebras of infinitely differentiable functions. J. Math. Mech., **11**, 797–809 (1962).