84. *Evaluation of Peirce's Axiom on Intermediate Kripke Models and its Application*

By Tsutomu HOSOI
Department of Information Sciences,
Science University of Tokyo

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We deal with the intermediate propositional logics. We suppose familiarity with the intermediate logics. For those not mentioned here explicitly, we refer to our Survey [1].

The so-called Peirce's axiom P is $a \supset b \supset a \supset a$, where parentheses are omitted by assuming the association from the left. This axiom was modified by Nagata into a sequence of axioms as follows:

Definition 1. $P_1(a_0, a_1) = a_1 \supset a_0 \supset a_1 \supset a_1$,

$$P_n(a_0, \cdots, a_n) = a_n \supset P_{n-1}(a_0, \cdots, a_{n-1}) \supset a_n \supset a_n.$$

This sequence is often used in the study of intermediate logics as a strong and convenient tool. The evaluation of axioms from this sequence is sometimes treated in literatures but the treatment seems not to be complete.

Here we give a complete treatment of the evaluation and its application for the axiomatization of infinite models.

As models for intermediate propositional logics, we take up the so-called Kripke model, which was modified and renamed as POS model by Ono (see Ono [2]). Further, we treat only finite models.

Let M be a POS model (usually, with the minimum element). We define the condition $\mathrm{C}(W, \alpha)$ for the M-valuation W and the element α of M as follows:

Condition.

$\mathrm{C}(W, \alpha)$: *$W(a, \alpha) = \boldsymbol{f}$ and, for any $\beta > \alpha$, $W(a, \beta) = \boldsymbol{t}$.*
And there exists $\beta > \alpha$ such that $W(b, \beta) = \boldsymbol{f}$.

The main result of this note is the following

Theorem 2. *$W(P, \alpha) = \boldsymbol{f}$ if and only if there exists $\alpha_0 \geq \alpha$ such that* $\mathrm{C}(W, \alpha_0)$.

We prove the theorem through the two lemmas as below.

Lemma 3. $\mathrm{C}(W, \alpha_0)$ *implies $W(P, \alpha_0) = \boldsymbol{f}$.*

Proof. By the hypothesis, we have $W(a, \beta) = \boldsymbol{t}$ for any $\beta > \alpha_0$. Further, there exists $\beta > \alpha_0$ such that $W(b, \beta) = \boldsymbol{f}$. Hence we have $W(a \supset b, \alpha_0) = \boldsymbol{f}$. So we have, for any $\gamma \geq \alpha_0$, $W(a \supset b, \gamma) = \boldsymbol{f}$ or $W(a, \gamma) = \boldsymbol{t}$. Hence we have $W(a \supset b \supset a, \alpha_0) = \boldsymbol{t}$. Finally, from $W(a \supset b \supset a, \alpha_0)$

$=\boldsymbol{t}$ and $W(a, \alpha_0)=\boldsymbol{f}$, we have $W(P, \alpha_0)=\boldsymbol{f}$.

Remark that the above proof does not use the finiteness of $\boldsymbol{M}$.

Lemma 4. *If $W(P, \alpha)=\boldsymbol{f}$, then there exists $\alpha_0 \geq \alpha$ such that* $\mathrm{C}(W, \alpha_0)$.

Proof. It is obvious that $W(P, \beta)=\boldsymbol{t}$ for any maximal $\beta \in \boldsymbol{M}$. Since $W(P, \alpha)=\boldsymbol{f}$, there exists non-maximal $\alpha_0 \geq \alpha$ such that $W(P, \alpha_0)=\boldsymbol{f}$ and, for any $\beta > \alpha_0$, $W(P, \beta)=\boldsymbol{t}$. From this, we have that $W(a \supset b \supset a, \alpha_0)=\boldsymbol{t}$ and $W(a, \alpha_0)=\boldsymbol{f}$. $W(a \supset b \supset a, \alpha_0)=\boldsymbol{t}$ implies that, for any $\beta > \alpha_0$, $W(a \supset b \supset a, \beta)=\boldsymbol{t}$. From this and $W(P, \beta)=\boldsymbol{t}$, we have $W(a, \beta)=\boldsymbol{t}$. Since $W(a, \alpha_0)=\boldsymbol{t}$ comes from $W(a \supset b, \alpha_0)=\boldsymbol{t}$ and $W(a \supset b \supset a, \alpha_0)=\boldsymbol{t}$, it must be that $W(a \supset b, \alpha_0)=\boldsymbol{f}$. Hence there exists $\beta > \alpha_0$ such that $W(a, \beta)=\boldsymbol{t}$ and $W(b, \beta)=\boldsymbol{f}$. So the condition $\mathrm{C}(W, \alpha_0)$ is satisfied.

Corollary 5. *$W(P_n(a_0, \cdots, a_n), \alpha)=\boldsymbol{f}$ if and only if there exists a chain $\alpha_0 > \alpha_1 > \cdots > \alpha_{n-1} > \alpha_n \geq \alpha$ in $\boldsymbol{M}$ such that*

(1) $W(a_0, \alpha_0)=\boldsymbol{f}$,

(2) $W(a_i, \alpha_i)=\boldsymbol{f}$ *and, for any* $\beta > \alpha_i$, $W(a_i, \beta)=\boldsymbol{t}$ $(i=1, 2, \cdots, n)$.

This corollary is a refinement of Lemma 3.4 in Ono [2].

Now we define two kinds of axioms.

Definition 6. $A_n = Z(a_n, b) \vee P_n$,

$B_n = Z(a_0, b) \vee P_n$,

where $Z(a, b)=(a \supset b) \vee (b \supset a)$ and $P_n = P_n(a_0, \cdots, a_n)$ as usual.

Let $\mathcal{F}_n$ be the set of all the finite POS models with the height $\leq n$ and $\mathcal{G}_n$ be the set of all the finite and irreducible POS models with the height $\leq n$.

Our next objective is to prove

Theorem 7. (1) $\boldsymbol{LJ}+A_n = \bigcap_{\substack{1 \leq k < \omega \\ N \in \mathcal{F}_n}} (\boldsymbol{S}_k \uparrow \boldsymbol{N})$,

(2) $\boldsymbol{LJ}+B_n = \bigcap_{\substack{0 \leq k < \omega \\ N \in \mathcal{G}_n}} (\boldsymbol{N} \uparrow \boldsymbol{S}_k)$,

where $\boldsymbol{N} \uparrow \boldsymbol{S}_0$ means $\boldsymbol{N}$ itself.

Proof. As is easily seen, the axioms A_n and B_n can be rewritten as I axioms. Hence the logics $\boldsymbol{LJ}+A_n$ and $\boldsymbol{LJ}+B_n$ have the finite model property.

Validity. (1) Let $\boldsymbol{M}$ be a model of the form $\boldsymbol{S}_k \uparrow \boldsymbol{N}$ where $1 \leq k < \omega$ and $\boldsymbol{N} \in \mathcal{F}_n$. Suppose that there exists an $\boldsymbol{M}$-valuation W and an element α of $\boldsymbol{M}$ such that $W(A_n, \alpha)=\boldsymbol{f}$. Since $W(P_n, \alpha)=\boldsymbol{f}$, we have that $\alpha \in \boldsymbol{S}_k$ by Corollary 5. Then, it must be that $W(a_n, \beta)=\boldsymbol{t}$ for any $\beta \in \boldsymbol{N}$, which implies that $W(Z(a_n, b), \alpha)=\boldsymbol{t}$. Hence A_n is valid in $\boldsymbol{M}$.

(2) Let $\boldsymbol{M}$ be a model of the form $\boldsymbol{N} \uparrow \boldsymbol{S}_k$ where $0 \leq k < \omega$ and $\boldsymbol{N} \in \mathcal{G}_n$. We suppose that $k \geq 1$ since the case $k=0$ is obvious. Suppose that $W(B_n, \alpha)=\boldsymbol{f}$. Since $W(Z(a_0, b), \alpha)=\boldsymbol{f}$, there exist $\beta_1, \beta_2 \geq \alpha$ such that $W(a_0, \beta_1)=\boldsymbol{t}$, $W(a_0, \beta_2)=\boldsymbol{f}$, $W(b, \beta_1)=\boldsymbol{f}$ and $W(b, \beta_2)=\boldsymbol{t}$. This implies

that $\beta_1, \beta_2, \alpha \in N$. Since $\alpha \in N \in \mathcal{G}_n$ and $W(P_n, \alpha)=f$, it must be that $W(a_0, \beta)=f$ for any $\beta \in N$ by Corollary 5. This is a contradiction. Hence B_n is valid in M.

Completeness. (1) Let M be a finite and irreducible POS model in which A_n is valid. Since M is irreducible, M can be written as $S_k \uparrow N$ with some $k \geq 1$ and some N. Let k be the possible biggest one. If N gets to be empty, then M is linear and of the required form. Suppose that N is not empty. Then N has at least two minimal elements. Suppose that $N \notin \mathcal{F}_n$. Then there exists an M-valuation W such that $W(P_n, \alpha)=f$ with a minimal element α. By this M-valuation, $W(a_n, \alpha)=f$. Further, we can suppose that $W(b, \alpha)=t$ and that, for any $\beta > \alpha$, $W(a_n, \beta)=t$ by Corollary 5. Let α' be another minimal element of N. Suppose that $W(a_n, \alpha')=t$ and $W(b, \alpha')=f$. This supposition does not interfere with $W(P_n, \alpha)=f$. Then we have $W(A_n, \gamma)=f$ for any $\gamma \in S_k$. This contradicts with the assumption that A_n is valid in M. Hence M is of the given form.

(2) Let M be a finite and irreducible POS model in which B_n is valid. M can be written as $N \uparrow S_k$ with some $k \geq 0$ and some N. Let k be the possible biggest one. If N gets to be empty, then M is of the required form. Suppose that $N \notin \mathcal{G}_n$. N has at least two maximal elements. There exists a chain $\alpha_0 > \alpha_1 > \cdots > \alpha_n$ in N with a maximal α_0 and the minimum α_n. Let α' be another maximal element of N. Let W be an M-valuation such that

(1) $W(a_0, \alpha_0)=f$, $W(b, \alpha_0)=t$,

(2) $W(a_i, \alpha_i)=f$ and, for any $\beta > \alpha_i$, $W(a_i, \beta)=t$ $(i=1, 2, \cdots, n)$,

(3) $W(a_0, \alpha')=t$, $W(b, \alpha')=f$.

Then, we have $W(B_n, \alpha_n)=f$, which is contradictory. Hence M is of the given form.

Corollary 8.
$$LJ+A_n=\bigcap_{N \in \mathcal{F}_n} (S_\omega \uparrow N),$$
$$LJ+B_n=LP_n \cap \bigcap_{N \in \mathcal{G}_n} (N \uparrow S_\omega),$$
$$LP_{m+n}+A_n=\bigcap_{N \in \mathcal{F}_n} (S_m \uparrow N),$$
$$LP_{m+n}+B_n=LP_n \cap \bigcap_{N \in \mathcal{G}_n} (N \uparrow S_m).$$

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References

[1] T. Hosoi and H. Ono: Intermediate Propositional Logics (A Survey). J. Tsuda College, **5**, 67–82 (1973).

[2] H. Ono: Kripke Models and Intermediate Logics. Publ. RIMS, Kyoto Univ., **6**, 461–476 (1970/71).