77. *Remarks on Hadamard's Variation of Eigenvalues of the Laplacian*

By Shin OZAWA
Department of Mathematics, University of Tokyo

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§ 1. **Introduction.** The study in this note is a continuation of our previous paper [6]. Let Ω be a bounded domain in $\mathbf{R}^n$ $(n\geq 2)$ with C^∞ boundary γ. Let $\rho(x)$ be a smooth function on and ν_x be the exterior unit normal vector at $x\in\gamma$. For any sufficiently small $\varepsilon\geq 0$, let Ω_ε be the bounded domain whose boundary γ_ε is defined by $\gamma_\varepsilon=\{x+\varepsilon\rho(x)\nu_x\,;\ x\in\gamma\}$. Let $U_\varepsilon(x,y,t)$ be the Green kernel of the heat equation in Ω_ε with the Dirichlet boundary condition on γ_ε. Let $T_r(t\,;\varepsilon)$ be the trace of U_ε on Ω_ε. When t tends to zero, we have the asymptotic expansion $T_r(t\,;\varepsilon)\sim\sum_{j=0}^{\infty}a_{n-j}(\varepsilon)(\sqrt{t})^{-n+j}$ which was given by Minakshisundarum-Pleijel [5]. In [6], the author gave the asymptotic expansion $\delta T_r(t)\sim\sum_{j=0}^{\infty}b_{n-j}(\sqrt{t})^{-n+j}$ near $t=0$ of the variational term $\delta T_r(t)$ of the trace which was defined by $\delta T_r(t)=\lim_{\varepsilon\to 0}\varepsilon^{-1}(T_r(t\,;\varepsilon)-T_r(t\,;0))$. We proposed the following problem $(\mathrm{E})_k^n$ in [6] and gave an affirmative answer for the case $k=0$.

Problem $(\mathrm{E})_k^n$. *Can we say that the following is valid?*

$$(\mathrm{E})_k^n \qquad b_{n-k}=\lim_{\varepsilon\to 0}\varepsilon^{-1}(a_{n-k}(\varepsilon)-a_{n-k}(0)).$$

In this paper, we shall prove the following

Theorem 1. $(\mathrm{E})_1^n$ *is valid for any* $n\geq 2$.

The aims of this note are verification of Theorem 1 and an application of Theorem 1 to some eigenvalue problem which will be stated in this section.

We now mention the following

Problem (Q). *Characterize the bounded domain Ω with smooth boundary γ having the following property.*

(I) *For any $\rho(z)\in C^\infty(\gamma)$ such that $\int_\gamma\rho(z)d\sigma_z=0$, we have $\delta\lambda_1=0$, where $\delta\lambda_1$ is the variational term of the first eigenvalue $\lambda_1<0$ of the Laplacian with the Dirichlet condition. Here $d\sigma_z$ denotes the surface element of γ at z.*

The condition $\int_\gamma\rho(z)d\sigma_z=0$ means that the perturbation of domain we considered preserves the volume of domains infinitesimally. We call the domain Ω satisfying the property (I) stationary domain. By the theorem of Hadamard-Garabedian-Schiffer we conclude that Ω is stationary if and only if

$$(1)\qquad \begin{cases} \Delta\varphi_1(x)-\lambda_1\varphi_1(x)=0 & \text{in } \Omega \\ \varphi_1(x)=0 & \text{on } \gamma \\ \dfrac{\partial\varphi_1}{\partial\nu_x}(x)=C & \text{on } \gamma, \end{cases}$$

where φ_1 is the normalized eigenfunction of the Laplacian with the Dirichlet condition and C is a constant. See [4] and [2].

The classical Faber-Krahn theorem states that the two dimensional domain Ω with the fixed area A which maximize the first eigenvalue $\lambda_1(\Omega)<0$ of the Laplacian with the Dirichlet condition is the disk of radius $(A\pi^{-1})^{1/2}$. In connection with this fact, we conjecture that any n-dimensional stationary domain is an open n-ball. For the n-ball with radius p, we have (1), so our problem may be restated as follows.

(Q)$^{\text{bis}}$: Are the following two statements equivalent?

(Q)$_1$: Ω is an open n-ball.

(Q)$_2$: Ω is an n-dimensional stationary domain.

Even if $n=2$, the implication from (Q)$_2$ to (Q)$_1$ is not trivial. It should be remarked that in general there is a gap between the concept of stationary value and that of maximum value in variational problem. In this paper, we give a partial answer to the problem (Q)$^{\text{bis}}$ by using Theorem 1.

The domain satisfying the following property will be called to be T-stationary:

(II) For any $\rho(z)\in C^\infty(\gamma)$ such that $\int_\gamma \rho(z)d\sigma_z=0$ and for any $t>0$, we have $\delta T_\gamma(t)=0$.

By Theorem 3 in [6], we know that any T-stationary domain is stationary. We have the following

Theorem 2. *Assume that Ω is T-stationary. Then for $n=2$ Ω is a disk, and for $n\geq 3$ every component of γ is a hypersurface of constant mean curvature.*

In § 2, we consider some geometry of hypersurfaces and the asymptotics of heat equation. In § 3, we give proofs of Theorems 1 and 2.

§ 2. Geometry of hypersurfaces and asymptotic expansion. We proved in [9] that

$$(2)\qquad \delta T_\gamma(t)=t\int_\gamma \frac{\partial^2 U(y,w,t)}{\partial\nu_y\partial\nu_w}\bigg|_{y=w=z}\rho(z)d\sigma_z.$$

Here we abbreviate $U_0(x,y,t)$ in § 1 as $U(x,y,t)$. In [8], we proved the following proposition by using the hard calculus of pseudo-differential operators such as [10].

Proposition 1. *If t tends to zero, then*

$$t\cdot\frac{\partial^2 U(y,w,t)}{\partial\nu_y\partial\nu_w}\bigg|_{y=w=z}\sim\sum_{k=0}^{\infty}B_{n-k}(z)t^{-(n-k)/2} \tag{3}$$

for any $z\in\gamma$. *Here* $B_{n-k}(z)\in C^\infty(\gamma)$.

We fix an arbitrary point z on γ. Without loss of generalities we can assume that z is the origin of $\mathbf{R}^n$. We take the z_n-axis as being coincident with the direction of the interior normal and the hyperplane $z_n=0$ as being coincident with the tangent hyperplane of γ at z. We take an orthonormal coordinate system $z'=(z_1,\cdots,z_{n-1})$ on this hyperplane. Then γ can be locally written as $z_n=\theta(z_1,\cdots,z_{n-1})$. Here $\theta\in C^\infty(\mathbf{R}^{n-1})$ and it has the Taylor expansion $\theta(z_1,\cdots,z_{n-1})=\sum_{|\alpha|\geq 2}^{\infty}\omega_\alpha z'^\alpha$. In [8], we also proved the following proposition by using the calculus of pseudo-differential operators. See also [7].

Proposition 2. *For any k, there exists a constant w(k) such that* $B_{n-k}(z)$ *is a polynomial of the variables* $\{\omega_\alpha\}_{2\leq|\alpha|\leq w(k)}$ *whose coefficients depend only on k.*

We write $B_{n-k}(z)=B_{n-k}(\{\omega_\alpha\})$. Following the idea of [1] and [3], we use the notion of the weight of polynomials. We give the weight $|\alpha|-1$ to the variable ω_α, and we give the weight $\sum_{j=1}^{s}\beta_j(|\alpha_j|-1)$ to the monomial $\prod_{j=1}^{S}(\omega_{\alpha_j})^{\beta_j}$. If P is the sum of the monomials of the same weight, we say that P is homogeneous. Now we prove the following

Theorem 3. $B_{n-k}(z)\equiv B_{n-k}(\{\omega_\alpha\})$ *is the homogeneous polynomial of the weight k.*

Proof. Fix $q>0$. We take a new coordinate system $\tilde{z}$ as follows; $\tilde{z}_j=qz_j$ $(j=1,\cdots,n)$. Then γ can be locally written as $\tilde{z}_n=\tilde{\theta}(\tilde{z}_1,\cdots,\tilde{z}_{n-1})=\tilde{\theta}(\tilde{z}')=\sum_{|\alpha|\geq 2}\tilde{\omega}_\alpha\tilde{z}'^\alpha$. We have the relation

$$\tilde{\omega}_\alpha=q^{-(|\alpha|-1)}\omega_\alpha. \tag{4}$$

On the other hand, the fundamental solution $\tilde{U}(\tilde{x},\tilde{y},t)$ of the heat equation $\frac{\partial}{\partial t}-\Delta_{\tilde{x}}$ with the Dirichlet boundary condition in new coordinates is related to $U(x,y,t)$ by

$$\tilde{U}(\tilde{x},\tilde{y},t)=q^{-n}U(x,y,q^{-2}t), \tag{5}$$

where $\Delta_{\tilde{x}}$ is the Laplacian in x-coordinates. It is easy to see

$$\frac{\partial^2 U(\tilde{y},\tilde{w},t)}{\partial\nu_{\tilde{y}}\partial\nu_{\tilde{w}}}\bigg|_{\tilde{y}=\tilde{w}=\tilde{z}}=q^{-n-2}\frac{\partial^2 U(y,w,q^{-2}t)}{\partial\nu_y\partial\nu_w}\bigg|_{y=w=z}, \tag{6}$$

where $\frac{\partial}{\partial\nu_z}$ is the derivative along the exterior normal vector. We compare the asymptotic expansion of both sides of (6). Then we get

$$B_{n-k}(\{\tilde{\omega}_\alpha\})=q^{-k}B_{n-k}(\{\omega_\alpha\}). \tag{7}$$

By (4) and (7), we obtain Theorem 3.

We transform $z'=(z_1, \cdots, z_{n-1})$ to another orthonormal basis $\hat{z}'=(\hat{z}_1, \cdots, \hat{z}_{n-1})=(z_1, \cdots, z_{n-1})V$. Here $V\in 0(n-1, \mathrm{R})$. Then γ can be locally written as $z_n=\hat{\theta}(\hat{z}_1, \cdots, \hat{z}_{n-1})=\sum_{|\alpha|\geq 2}\hat{\omega}_\alpha \hat{z}'^\alpha$. We can represent $\{\hat{\omega}_\alpha\}$ explicitly by $\{\omega_\alpha\}$ and $V\in 0(n-1, \mathrm{R})$. It should be remarked that

$$B_{n-k}(\{\hat{\omega}_\alpha\})=q^{-k}B_{n-k}(\{\omega_\alpha\}), \tag{8}$$

since both sides are equal to $B_{n-k}(z)$ which does not depend on the choice of orthonormal basis on the tangent hyperplane of γ at z.

We have the following

Proposition 3. *$B_n(z)=e_n$ on γ, and $B_{n-1}(z)=\beta H_1(z)$ on γ. Here $H_1(z)$ denotes the first mean curvature of γ at z, and e_n, β are constants depending only on n.*

Proof. Since $B_n(z)$ is homogeneous of weight 0, it must be a constant. Next we study $B_{n-1}(z)$. If we transform z' to $\hat{z}'=z'V$, where $V\in 0(n-1, \mathrm{R})$, then the following relation holds.

$$\hat{\omega}_{kh}=\sum_{i,j=1}^{n-1} V_{ki}\omega_{ij}V_{ih}. \tag{9}$$

The above relation can be rewritten as

$$\hat{\omega}=V\omega V^{-1}, \tag{10}$$

where ω and $\hat{\omega}$ is the matrix with the components $\{\omega_{ij}\}$ and $\{\hat{\omega}_{kh}\}$ respectively. Since $B_{n-1}(z)$ is homogeneous of weight 1, it must be written as $B_{n-1}(z)=\sum_{i,j=1}^{n-1} s_{ij}\omega_{ij}$, where s_{ij} is a constant which depend only on n. So we have also $B_{n-1}(z)=\sum_{i,j=1}^{n-1} s_{ij}\hat{\omega}_{ij}$ for any $\hat{\omega}_{ij}$ with respect to any other orthonormal coordinates $\hat{z}'$. Finally by the theory of orthogonal invariants such as [11], we conclude that $s_{ij}=\tau\delta_{ij}$ where τ is a constant and δ_{ij} is the Kronecker delta. It is well known that $H_1(z)=2(n-1)^{-1}\sum_{i=1}^{n-1}\omega_{ii}$. The proof is over.

§3. Proofs of Theorems 1 and 2. The coefficients $a_n(\varepsilon)$ and $a_{n-1}(\varepsilon)$ in the asymptotic expansion of $T_\gamma(t;\varepsilon)$ are represented as $a_n(\varepsilon)=C_n|\Omega_\varepsilon|$, $a_{n-1}(\varepsilon)=C_{n-1}|\gamma_\varepsilon|$. Here C_n and C_{n-1} are non zero constants which depend only on n. And here $|\Omega_\varepsilon|$ denotes the volume of Ω_ε and $|\gamma_\varepsilon|$ denotes the area of γ_ε. Therefore we have to show

$$b_n=C_n\int_\gamma \rho(z)d\sigma_z \tag{11}$$

and

$$b_{n-1}=(n-1)C_{n-1}\int_\gamma H_1(z)\rho(z)d\sigma_z. \tag{12}$$

By Theorems 1 and 2 in [6], we have

$$b_{n-k}=\int_\gamma B_{n-k}(z)\rho(z)d\sigma_z.$$

So we can restate $(\mathrm{E})_k^n$ $(k=0, 1)$ as follows.

$(\mathrm{E})_0^n$ $$e_n = C_n,$$
$(\mathrm{E})_1^n$ $$e_{n-1}(z) = (n-1)C_{n-1}H_1(z).$$

Let B_R be the open ball with radius R centered at the origin. Let $0 \geq \lambda_1(R) \geq \lambda_2(R) \geq \cdots$ be the eigenvalues of the Laplacian with the Dirichlet boundary condition at ∂B_R. We arrange them according to their multiplicities. We have $\lambda_i(R) = \lambda_i(1)R^{-2}$. Put $T_\gamma(t\,|\,B_R) = \sum_{j=1}^{\infty} e^{\lambda_j(R)t}$. And put $\rho(z) = 1$ on $\gamma = \partial B_R$. Then we have

$$\delta T_\gamma(t) = \frac{\partial}{\partial R} T_\gamma(t\,|\,B_R) = -2tR^{-1}\frac{\partial}{\partial t} T_\gamma(t\,|\,B_R).$$

On the other hand, by the calculus of pseudo-differential operators we can get the following proposition. We need nontrivial calculations to prove it. See [8].

Proposition 4. *When t tends to zero, $\frac{\partial}{\partial t} T_\gamma(t\,|\,B_R)$ has the asymptotic expansion*

$$\frac{\partial}{\partial t} T_\gamma(t\,|\,B_R) \sim \sum_{k=0}^{\infty} M_{n-k}(R) t^{-(n+2-k)/2}.$$

By Proposition 4, we can differentiate with respect to t the asymptotic expansion $T_\gamma(t\,|\,B_R) \sim \sum_{k=0}^{\infty} a_{n-k}(\sqrt{t})^{-n+k}$ term by term. Therefore we have

(13) $$\delta T_\gamma(t) \sim n a_n R^{-1}(\sqrt{t})^{-n} + \cdots + (n-k)a_{n-k}R^{-1}(\sqrt{t})^{-n+k} + \cdots.$$

On the other hand, we have

(14) $$\delta T_\gamma(t) \sim e_n\,|\gamma|(\sqrt{t})^{-n} + \beta \int_\gamma H_1(z)d\sigma_z \cdot (\sqrt{t})^{-n+1} + 0(t^{-n/2+1}).$$

By (13) and (14), we have $|\gamma|\,e_n = n a_n R^{-1}$ and $\beta\,|\gamma|\,R^{-1} = (n-1)a_{n-1}R^{-1}$. Since $a_n = C_n\,|B_R|$, $a_{n-1} = C_{n-1}\,|\partial B_R|$, we get Theorem 1.

By Theorem 1 we have the following

Corollary. *If t tends to zero, then*

(15) $$\delta T_\gamma(t) = C_n \int_\gamma \rho(z)d\sigma_z \cdot t^{-(n+2)/2} + (n-1)C_{n-1} \cdot \int_\gamma H_1(z)\rho(z)d\sigma_z \cdot t^{-(n+1)/2} + 0(t^{-n/2}),$$

where $C_n \neq 0$, $C_{n-1} \neq 0$.

Proof of Theorem 2. The proof of Theorem 2 is obvious for $n \geq 3$, since we have (15). Now we study the case $n=2$. By (15) we can conclude that γ consists of finite disjoint union of circles. Since we know that any T-stationary domain is stationary by Theorem 3 in [6], we have (1). Let $\tilde{\gamma}$ be the largest circle contained in γ. By the uniqueness theorem of Holmgren we conclude that the level set $\tilde{\gamma}(s) = \{z\,;\,\varphi_1(z) = s,\, z \in \Omega\}$ consists of the finite disjoint union of circles with the same center as $\tilde{\gamma}$. It is well known that $\varphi_1(z)$ does not take zero in

Ω. Summing up these facts we conclude that Ω must be the disk or the annulus. For the annulus $\{z \in \mathbf{R}^2; r_1<|z|<r_2\}$, $\varphi_1(z)$ can be calculated explicitly. If $r_1>0$, then $\varphi_1(z)$ with respect to this domain does not satisfy (1). The proof is over.

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