52. A Note on Almost-Primes in Short Intervals

By Yoichi MOTOHASHI
Department of Mathematics, College of Science
and Technology, Nihon University

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1. In this note we are concerned with the existence of P_2 (numbers having at most two prime factors) in almost all short intervals. The hitherto best result in this field is due to Heath-Brown [1], who has shown, using the weighted linear sieve, that for almost all x there exists a P_2 such that $x<P_2\leqq x+x^{1/11}$. Improving on this we shall prove, by an easy variant of Jutila's argument [2], the following result:

Theorem. *Let ε be an arbitrary small positive constant. Then for almost all x there exists a P_2 such that $x<P_2\leqq x+x^{\varepsilon}$.*

Before giving the proof we make some remarks. One may consider the possibility of reducing the length of intervals from x^{ε} to a power of $\log x$, which is desirable especially when Selberg's result [4] on primes in short intervals is taken into account. But the best result our argument can yield seems to be only $\exp(C(\log x)^{2/3}(\log\log x)^{4/3})$ with a large constant C, instead of x^{ε} in our theorem. Also one may ask how the situation is if we consider the least P_2 in almost all arithmetic progressions modulo a fixed integer. Because of the lack of a result on Dirichlet's L-functions comparable with Vinogradov's zero-free region for $\zeta(s)$ the Riemann zeta-function, our argument in the present note cannot be modified so as to be applicable to arithmetic progressions. Thus our result in [3] remains so far to be the best.

2. Now we enter into the proof. Let x be sufficiently large, and let us assume that $x>U, V>x^{1/3}$. And we put, denoting primes by p, p',

$$P(s)=\sum_{U<p\leqq 2U} p^{-s},$$

$$\Phi(y)=\sum_{\substack{y<pp'\leqq y(1+1/V)\\ U<p\leqq 2U}} \log p'.$$

Further we put

$$I_x=\frac{1}{x}\int_x^{2x}\left|\Phi(y)-\frac{y}{V}P(1)\right|^2 dy.$$

We have

$$\Phi(y)-\frac{y}{V}P(1)$$

$$=\frac{1}{2\pi i}\int_{\eta-iU}^{\eta+iU}\left\{G(s)-\frac{\zeta'}{\zeta}(s)\right\}P(s)\left(\left(1+\frac{1}{V}\right)^{s}-1\right)y^{s}\frac{ds}{s}$$
$$+O\left(\frac{x}{U}(\log x)^{3}\right).$$

In the above we have put

$$\eta=1-(\log x)^{-3/4},\qquad G(s)=\sum_{p}\frac{\log p}{p^{s}(1-p^{s})},$$

and have appealed to the well-known estimate

$$\frac{\zeta'}{\zeta}(\sigma+it)\ll\log|t|$$

for

$$\sigma\geqq 1-c(\log|t|)^{-2/3-\varepsilon},\qquad |t|\geqq 2.$$

Thus we get, after performing the integration over y,

$$I_x\ll\frac{x^{2\eta}}{V^{2}}(\log x)^{2}\int_{-U}^{U}\int_{-U}^{U}\frac{|P(\eta+iu)P(\eta+iv)|}{1+|u-v|}dudv+\left(\frac{x}{U}\right)^{2}(\log x)^{6}.$$

The double integral is obviously

$$\ll\int_{-U}^{U}|P(\eta+iu)|^{2}\,du\int_{-U}^{U}\frac{dv}{1+|u-v|}\ll U^{2(1-\eta)}\log U.$$

So we find

$$I_x\ll\left(\frac{x}{V}P(1)\right)^{2}\left\{(\log x)^{5}\left(\frac{U}{x}\right)^{2(1-\eta)}+\left(\frac{V}{U}\right)^{2}(\log x)^{8}\right\}.$$

Setting in this

$$V=x^{1-\varepsilon},\qquad U=V\exp((\log x)^{1/4}),$$

we get

$$I_x\ll\left(\frac{x}{V}P(1)\right)^{2}\exp(-(\log x)^{1/5}).$$

This ends the proof.

Added in Proof (June 18, 1979). Combining quite ingeniously the zero-density argument with our method, D. Wolke (Univ. Freiburg) has succeeded in reducing the length of the intervals of our theorem to a power of log x. See his forthcoming paper entitled "*Fast-Primzahlen in kurzen Intervallen*".

References

[1] D. R. Heath-Brown: Thesis. Cambridge Univ. (1977).

[2] M. Jutila: On numbers with a large prime factor. J. Indian Math. Soc., **37**, 43–53 (1973).

[3] Y. Motohashi: On almost primes in arithmetic progressions. J. Math. Soc. Japan, **28**, 363–383 (1976).

[4] A. Selberg: On the normal density of primes in small intervals and the difference between consecutive primes. Arch. Math. Naturvid., **47**, 87–105 (1943).