49. On the Boundary Behavior of Taylor Series of Regular Functions of Some Classes in the Unit Circle

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1. **Introduction.** In his previous papers ([3], [4]), the author introduced (C, k, α)-summation, by means of which Taylor series of the regular function of bounded type in $|z|<1$ can be summable on $|z|=1$. In this note, for the class wider than bounded type, he studies the convergence, the almost everywhere convergence and the mean convergence of this summation.

2. **Statement of results.** For the sake of completeness, we recall the definition of (C, k, α)-summation. Let $f(z)$ be a regular function in $|z|<1$:

$$f(z)=\sum_{n=0}^{\infty} a_n z^n.$$

For two constants $k, \alpha (k>-1, \alpha>O)$, we put

$$\frac{1}{(1-z)^{k+1}}\cdot \exp\left(\frac{\alpha}{1-z}\right)=\sum_{n=0}^{\infty} b_n(k, \alpha) z^n,$$

where

$$(1)\ b_n(k, \alpha)>O,\ (2)\ b_n(k, \alpha)\sim \frac{\exp(\alpha/2+2\sqrt{\alpha n})}{2\sqrt{\pi}\,\alpha^{1/4+k/2} n^{1/4-k/2}} \text{ as } n\to\infty,$$

and let

$$\frac{1}{(1-z)^{k+1}}\cdot \exp\left(\frac{\alpha}{1-z}\right)\sum_{n=0}^{\infty} a_n e^{in\theta} z^n=\sum_{n=0}^{\infty} S_n(k, \alpha, e^{i\theta})\cdot z^n.$$

If $C_n(k, \alpha, e^{i\theta})=S_n(k, \alpha, e^{i\theta})/b_n(k, \alpha)\to s$ as $n\to\infty$, we say that the series $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is summable (C, k, α) to s.

Our Theorem 1 reads as follows.

Theorem 1. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be a regular function in $|z|<1$ such that*

$$(2.1)\qquad \varlimsup_{r\to 1}(1-r)\cdot \log^+ M(r)=\delta<+\infty,$$

where $M(r)=\max_{|z|=r}|f(z)|$. Then the following propositions hold:

(A) *If $f(z)$ has the finite angular limit $f(e^{i\theta})$ at $z=e^{i\theta}$, then for any $\alpha>\delta$, $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is summable (C, k, α) to $f(e^{i\theta})$.*

$$\text{(B)}\qquad \rho^n\cdot \int_{-\pi}^{\pi} |C_n(k, \alpha, e^{i\theta})-f(\rho e^{i\theta})|\, d\theta=o(1) \text{ as } n\to\infty,$$

where $\rho=1-\sqrt{\alpha/n}$, $\alpha>\delta$.

We denote by N the class of functions $f(z)$ regular and bounded type in the unit circle. Then we have

$$A(f)=\lim_{r\to 1} 1/2\pi\cdot\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta<+\infty.$$

N^+ is the subclass of N of functions $f(z)$ satisfying

$$A(f)=\lim_{r\to 1} 1/2\pi\cdot\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta=1/2\pi\cdot\int_0^{2\pi} \log^+ |f(e^{i\theta})|\, d\theta<+\infty.$$

As its applications to the class of bounded type, we get two corollaries.

Corollary 1. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n \in N$ *in* $|z|<1$. *Put*

$$\beta=2A(f)=\lim_{r\to 1} 1/\pi\cdot\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta.$$

If $\beta<\alpha$, *then the following propositions hold*:

(A) $\sum_{n=0}^{\infty} a_n e^{in\theta}$ *is summable* (C, k, α) *to* $f(e^{i\theta})$ *a.e. on* $|z|=1$.

(B) $$\rho^n\cdot\int_{-\pi}^{\pi} |C_n(k, \alpha, e^{i\theta})-f(\rho e^{i\theta})|d\theta=o(1) \text{ as } n\to\infty,$$

where $\rho=1-\sqrt{\alpha/n}$.

Corollary 2. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n \in N^+$ *in* $|z|<1$. *Then for any* $\varepsilon>O$, *we have*

(A) $\sum_{n=0}^{\infty} a_n e^{in\theta}$ *is summable* (C, k, ε) *to* $f(e^{i\theta})$ *a.e. on* $|z|=1$.

(B) $$\rho^n\cdot\int_{-\pi}^{\pi} |C_n(k, \varepsilon, e^{i\theta})-f(\rho e^{i\theta})|\, d\theta=o(1) \text{ as } n\to\infty,$$

where $\rho=1-\sqrt{\varepsilon/n}$.

Remark 1. N. Yanagihara ([7, p. 332], [8]) has independently introduced the same summation as (C, k, α)-summation, and he proved Corollary 2(A) by the entirely different method. He also proved Corollary 1(A) for *sufficiently large* α ([8]), but it holds for *any* α *greater than* β.

In Theorem 1(A), under some additional conditions on the growth of $f(z)$, we can prove (C, k, δ) summability at $z=e^{i\theta}$ instead of (C, k, α) $(\alpha>\delta)$ summability. Here we remark that next inclusions hold; for $\delta<\alpha$,

(C, k, δ)-summation $\subset (C, k, \alpha)$-summation $\subset$ Abel summation.

Now we introduce

Definition. Let $f(z)$ be a regular function in $|z|<1$ such that

$$\overline{\lim_{r\to 1}}\, (1-r) \log^+ M(r)=\delta<+\infty,$$

where $M(r)=\max_{|z|=r} |f(z)|$. If there exists a constant $r_1 (O<r_1<1)$ such that

$$M(r)<\exp\left(\frac{\delta}{1-r}\right) \qquad \text{for } r_1\leqq r<1,$$

we say that $f(z)$ has the exact type δ.

Using Definition, we can prove

Theorem 2. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be a regular function of the exact type* $\delta (O<\delta<+\infty)$ *in* $|z|<1$. *Then the following propositions hold*:

(A) *Let $f(z)$ have the finite angular limit $f(e^{i\theta})$ at $z=e^{i\theta}$. Then $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is summable (C, k, δ) $(k>1/2)$ to $f(e^{i\theta})$, provided that*

$$\overline{\lim_{r\to 1}}\,(1-r)\log^+ M(r, \Delta, \theta)<\delta$$

for sufficiently small $\Delta>0$, where

$$M(r, \Delta, \theta)=\max_{|h|\leqq \Delta}\left|1/h\cdot\int_{\theta}^{\theta+h}|f(re^{i\phi})|\,d\phi\right|.$$

(B) $$\rho^n\cdot\int_{-\pi}^{\pi}|C_n(k, \delta, e^{i\theta})-f(\rho e^{i\theta})|\,d\theta=O(1) \text{ as } n\to\infty,$$

where $\rho=1-\sqrt{\delta/n}$.

From Theorem 2, we get the followings

Corollary 3. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n \in N$ in $|z|<1$, and let $\alpha=2A(f)>0$.*

(A) *If $f(z)$ has the finite angular limit $f(e^{i\theta})$ at $z=e^{i\theta}$, then $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is summable (C, k, α) $(k>1/2)$ to $f(e^{i\theta})$, provided that*

$$\overline{\lim_{r\to 1}}\,(1-r)\cdot\log^+ M(r, \Delta, \theta)<\alpha$$

for sufficiently small $\Delta>0$.

(B) $$\rho^n\cdot\int_{-\pi}^{\pi}|C_n(k, \alpha, e^{i\theta})-f(\rho e^{i\theta})|\,d\theta=O(1) \text{ as } n\to\infty,$$

where $\rho=1-\sqrt{\alpha/n}$.

Corollary 4. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n \in N$ in $|z|<1$, and let $\alpha=2A(f)>0$. If*

$$\overline{\lim_{r\to 1}}\,(1-r)\cdot\log^+ M(r, \varphi, \varphi')<\alpha,$$

where $M(r, \varphi, \varphi')=\max_{\varphi\leqq\theta\leqq\varphi'}|f(re^{i\theta})|$, then $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is summable (C, k, α) $(k>1/2)$ to $f(e^{i\theta})$ a.e. on the arc $C=\{e^{i\theta}: \varphi\leqq\theta\leqq\varphi'\}$.

Remark 2. In his previous papers ([3, p. 59], [4, p. 287]), the author proved Corollary 3(A) *under the superfluous condition* that $f(z)=f(e^{i\theta})+o(\sqrt{|z-e^{i\theta}|})$ as $z\to e^{i\theta}$ in Stolz domain with its vertex at $z=e^{i\theta}$.

3. Outline of the proof. Throughout this note, we use the following notations:

$$\eta=\pi/n, \quad \rho=\rho(\alpha)=1-\sqrt{\frac{\alpha}{n}} \quad (\alpha>0,\ n=1, 2, \cdots)$$

$$g(z)=g(z, \alpha)=\frac{1}{(1-z)^{k+1}}\cdot\exp\left(\frac{\alpha}{1-z}\right)(k>-1).$$

To establish our theorems, we need

Lemma 1. *The following equality holds:*

$$\begin{aligned} C_n(k, \alpha, e^{i\theta})-f(\rho e^{i\theta})&=O(e^{\sqrt{n\alpha}})/b_n(k, \alpha)\cdot I(n, \theta)\\ &=O(e^{\sqrt{n\alpha}})/b_n(k, \alpha)\cdot\{I_1(n, \theta)+I_2(n, \theta)\}, \end{aligned}$$

where

$$I(n, \theta)=\int_{-\pi}^{\pi}[f(\rho e^{i(\theta+\phi)})-f(\rho e^{i\theta})]\cdot g(\rho e^{i\phi})\cdot e^{-in\phi}d\phi,$$

$$I_1(n, \theta)=\int_{-\pi}^{\pi}[f(\rho e^{i(\theta+\phi)})-f(\rho e^{i\theta})]\cdot[g(\rho e^{i\phi})-g(\rho e^{i(\phi+\eta)})]\cdot e^{-in\phi}d\phi,$$

$$I_2(n,\theta)=\int_{-\pi}^{\pi}[f(\rho e^{i(\theta+\phi)})-f(\rho e^{i(\theta+\phi+\eta)})]\cdot g(\rho e^{i(\phi+\eta)})\cdot e^{-in\phi}d\phi.$$

This lemma is proved by next equalities:

$$f(ze^{i\theta})\cdot g(z)=\sum_{n=0}^{\infty}S_n(k,\alpha,e^{i\theta})\cdot z^n,\qquad e^{-in\phi}=1/2\cdot(e^{-in\phi}-e^{-in(\phi-\eta)}).$$

Lemma 2. *Let $f(z)$ be a function regular in $|z|<1$ and satisfying*

$$M(r)<\exp\left(\frac{\beta}{1-r}\right)\qquad \textit{for } O<r_0\leqq r<1,$$

where $M(r)=\max_{|z|=r}|f(z)|$, β: a positive constant. Then for any $\alpha(>O)$ and sufficiently large n, we have

$$\int_{-\pi}^{\pi}\max_{0\leqq\psi\leqq\eta}|f'(\rho e^{i(\theta+\psi)})|\,d\theta=O\left(n\cdot\exp\left(\sqrt{n\alpha}\cdot\frac{\beta}{\alpha}\right)\right),$$

where $\rho=\rho(\alpha)=1-\sqrt{\alpha/n}$.

This lemma is established by E. Goursat's theorem, Poisson's integral and G. H. Hardy's "Max" ([1, p. 114], [5, p. 186]).

Lemma 3. *We have the following estimate:*

$$e^{\sqrt{n\alpha}}\cdot\int_{-\pi}^{\pi}|g(\rho e^{i\phi})|\,d\phi=O(b_n(k,\alpha))\ \textit{as}\ n\to\infty$$

where $\rho=\rho(\alpha)=1-\sqrt{\alpha/n}$.

This lemma is proved by elementary but very delicate calculations.

Outline of the proof of Theorem 1. Without any loss of generality, we can assume that $\theta=O$. For the proof of Part (A), it sufficies to prove that $C_n(k,\alpha,1)-f(\rho)=o(1)$ as $n\to+\infty$. By (2.1), for any ε $(O<\varepsilon<\alpha-\delta)$, there exists $r_0(\varepsilon)$ such that

$$M(r)<\exp\left(\frac{\beta}{1-r}\right)\qquad\text{for } r_0(\varepsilon)\leqq r<1,\ \beta=\delta+\varepsilon<\alpha. \tag{3.1}$$

By Lemma 1,

$$C_n(k,\alpha,1)-f(\rho)=O(e^{\sqrt{n\alpha}})/b_n(k,\alpha)\cdot I(n,O). \tag{3.2}$$

We divide $I(n,O)$ into two parts:

$$I(n,O)=\int_0^{\pi}+\int_{-\pi}^{0}=I_1+I_2. \tag{3.3}$$

We further divide I_1 into two parts:

$$I_1=\int_A^B+\int_B^D=I_{1,1}+I_{1,2},$$

where $A: z=\rho$, $D: z=-\rho$, B: the first intersection point of the circle: $|z|=\rho$ and the half straight line: $z=1-te^{-i\theta}$ $(O\leqq t<+\infty, O<\theta<\pi/2)$. Since $f(z)$ has the finite angular limit $f(1)$ at $z=1$, on the arc $\widehat{AB}$ we have uniformly with respect to ϕ

$$f(\rho e^{i\phi})-f(\rho)=o(1)\ \text{as}\ n\to+\infty,$$

so that, by Lemma 3

$$|I_{1,1}|\leqq o(1)\cdot\int_A^B|g(\rho e^{i\phi})|\,d\phi<o(1)\cdot\int_{-\pi}^{\pi}|g(\rho e^{i\phi})|\,d\phi=o(e^{-\sqrt{n\alpha}}\cdot b_n(k,\alpha)).$$

On the $\widehat{BD}$, we have

$$\frac{1}{|1-\rho e^{i\phi}|} \leqq \frac{1}{BE} = \sqrt{\frac{n}{\alpha}} \cdot \cos\theta \cdot (1+o(1))$$

for sufficiently large n, where $E : z=1$, so that by (3.1)

$$|I_{1,2}| < 2\pi \cdot e^{\beta/(1-\rho)} \cdot \left(\frac{1}{1-\rho}\right)^{k+1} \cdot \exp\left(\sqrt{n\alpha} \cdot \cos\theta \cdot (1+o(1))\right)$$

$$= O(e^{-\sqrt{n\alpha}} b_n(k,\alpha)) \cdot n^{3/4} \cdot \exp\left(\sqrt{n\alpha} \cdot \left(-1 + \frac{\beta}{\alpha} + \cos\theta \cdot (1+o(1))\right)\right).$$

If θ is sufficiently near $\pi/2$, we have

$$-1 + \frac{\beta}{\alpha} + \cos\theta \cdot (1+o(1)) < 0.$$

Hence

$$|I_{1,2}| = o(e^{-\sqrt{n\alpha}} \cdot b_n(k,\alpha)),$$

so that $|I_1| = o(e^{-\sqrt{n\alpha}} \cdot b_n(k,\alpha))$. Similarly $|I_2| = o(\mathrm{e}^{-\sqrt{n\alpha}} \cdot b_n(k,\alpha))$. Therefore, by (3.2) and (3.3)

$$C_n(k,\alpha,1) - f(\rho) = o(1) \text{ as } n \to +\infty,$$

which proves Part (A).

By Lemma 1, we have

(3.4) $$|C_n(k,\alpha,e^{i\theta}) - f(\rho e^{i\theta})| \leqq O(e^{\sqrt{n\alpha}})/b_n(k,\alpha) \cdot \{I_n^{(1)}(\theta) + I_n^{(2)}(\theta)\},$$

where

$$I_n^{(1)}(\theta) = \int_{-\pi}^{\pi} |f(\rho e^{i(\theta+\phi)}) - f(\rho e^{i\theta})| \cdot |g(\rho e^{i\phi}) - g(\rho e^{i(\phi+\eta)})| \, d\phi,$$

$$I_n^{(2)}(\theta) = \int_{-\pi}^{\pi} |f(\rho e^{i(\theta+\phi)}) - f(\rho e^{i(\theta+\phi+\eta)})| \cdot |g(\rho e^{i(\phi+\eta)})| \, d\phi.$$

By (3.1)

$$\int_{-\pi}^{\pi} I_n^{(1)}(\theta) d\theta < 8\pi \cdot \exp\left(\sqrt{n\alpha} \cdot \frac{\beta}{\alpha}\right) \cdot \int_{-\pi}^{\pi} |g(\rho e^{i\phi})| \, d\phi.$$

By Lemma 2,

$$\int_{-\pi}^{\pi} I_n^{(2)}(\theta) d\theta = O\left(\exp\left(\sqrt{n\alpha} \cdot \frac{\beta}{\alpha}\right)\right) \cdot \int_{-\pi}^{\pi} |g(\rho e^{i\phi})| \, d\phi.$$

Hence, by (3.4)

$$\int_{-\pi}^{\pi} |C_n(k,\alpha,e^{i\theta}) - f(\rho e^{i\theta})| \, d\theta \leqq O(e^{\sqrt{n\alpha}})/b_n(k,\alpha) \cdot \int_{-\pi}^{\pi} |g(\rho e^{i\phi})| \, d\phi \times \exp\left(\sqrt{n\alpha} \cdot \frac{\beta}{\alpha}\right),$$

so that by Lemma 3,

$$\int_{-\pi}^{\pi} |C_n(k,\alpha,e^{i\theta}) - f(\rho e^{i\theta})| \, d\theta = O\left(\exp\left(\sqrt{n\alpha} \cdot \frac{\beta}{\alpha}\right)\right)$$

$$= \exp\left(\sqrt{n\alpha}\right) O\left(\exp\left(\sqrt{n\alpha}\left(-1 + \frac{\beta}{\alpha}\right)\right)\right).$$

Since $-1 + \beta/\alpha < O$,

$$\int_{-\pi}^{\pi} |C_n(k,\alpha,e^{i\theta}) - f(\rho e^{i\theta})| \, d\theta = o(\exp(\sqrt{n\alpha})) \text{ as } n \to +\infty,$$

which proves Part (B), taking account of $\rho^{-n} = \exp(\sqrt{n\alpha} + \alpha/2 + o(1))$.

Proof of Corollary 1. since $f(z)\in N$, the following properties hold:

(1) $f(z)$ has the finite angular limit a.e. on $|z|=1$,

(2) $M(r)<\exp(\beta/(1-r))$ for $0<r<1$ ([2, p. 57]), so that

$$\overline{\lim_{r\to 1}}(1-r)\cdot\log^+ M(r)=\delta\leqq\beta<\alpha.$$

Hence, Corollary 1 follows immediately from Theorem 1.

Proof of Corollary 2. By $f(z)\in N^+$, we have

(1) $f(z)$ has the finite angular limit a.e. on $|z|=1$,

(2) $\overline{\lim_{r\to 1}}(1-r)\cdot\log^+ M(r)=0$ ([6, p. 39]), so that Corollary 2 is an immediate consequence of Theorem 1.

Theorem 2 is also proved by the arguments which are similar to Theorem 1, but more delicate.

Proof of Corollary 3. Since $f(z)\in N$, we have

$$M(r)<\exp\left(\frac{\alpha}{1-r}\right) \qquad \text{for } 0<r<1 \text{ ([2, p. 57])}, \tag{3.5}$$

so that

$$\overline{\lim_{r\to 1}}(1-r)\cdot\log^+ M(r)=\delta\leqq\alpha.$$

In the case $\delta<\alpha$, by Theorem 1, Corollary 3 holds evidently. In the case $\delta=\alpha$, by (3.5) $f(z)$ has the exact type δ. Hence, by Theorem 2, Corollary 3 is proved.

Corollary 4 is an immediate consequence of Corollary 3. More detailed proof will be published elsewhere in near future.

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