25. On Zariski Problem

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In this note we generalize a result of Zariski [8, § 7]. As an application, using the theory of Miyanishi [5], [6], we prove the following

Theorem. *Let S be a surface defined over a field k of characteristic zero such that $S\times A^1\cong A^3$. Then $S\cong A^2$.*

Namely the so-called Zariski problem is solved in the affirmative way. Our method of proof will work also in positive characteristic cases provided that there is a sufficiently powerful analogue of the theory of Iitaka [1], [2]. It should be emphasized that the theory of Miyanishi plays a very important role in our proof.

§ 1. Zariski decomposition of pseudo effective line bundles.

Let S be a complete non-singular surface defined over an algebraically closed field k of any characteristic. *Prime divisor* means an irreducible reduced curve on S.

(1.1) A linear combination of prime divisors with coefficients in the rational number field $\boldsymbol{Q}$ is called a *$\boldsymbol{Q}$-divisor*. A $\boldsymbol{Q}$-divisor is said to be *effective* if each coefficient is non-negative.

(1.2) An element of $Pic(S)\otimes\boldsymbol{Q}$ is called a *$\boldsymbol{Q}$-line bundle*. Any $\boldsymbol{Q}$-divisor D defines naturally a $\boldsymbol{Q}$-line bundle, which is denoted by D by abuse of notation. For any $\boldsymbol{Q}$-line bundles F_1 and F_2, we define the intersection number $F_1F_2\in\boldsymbol{Q}$ in the obvious way.

(1.3) A $\boldsymbol{Q}$-line bundle H is said to be *semi-positive* if $HC\geqq 0$ for any prime divisor C. Then, obviously, $HE\geqq 0$ for any effective $\boldsymbol{Q}$-divisor E.

(1.4) **Lemma.** *Let H be a semipositive $\boldsymbol{Q}$-line bundle and let E be an effective $\boldsymbol{Q}$-divisor. If $(H+E)C_i\geqq 0$ for each prime component C_i of E, then $(H+E)$ is semipositive.*

Proof is easy.

(1.5) A $\boldsymbol{Q}$-line bundle L is said to be *pseudo effective* if $LH\geqq 0$ for any semipositive $\boldsymbol{Q}$-line bundle H. Clearly any effective $\boldsymbol{Q}$-divisor is pseudo effective.

(1.6) Let $C_1, \cdots, C_q$ be prime divisors. By $V(C_1, \cdots, C_q)$ we denote the $\boldsymbol{Q}$-vector space of $\boldsymbol{Q}$-divisors generated by $C_1, \cdots, C_q$. $I(C_1, \cdots, C_q)$ denotes the quadratic form on $V(C_1, \cdots, C_q)$ defined by the self intersection number.

(1.7) **Lemma.** *Let $C_1, \cdots, C_q$ be prime divisors such that $I(C_1, \cdots, C_q)$ is negative definite. Let $X \in V(C_1, \cdots, C_q)$ and suppose that $XC_i \leqq 0$ for any $i=1, \cdots, q$. Then X is effective.*

For a proof, see Zariski [8, p. 588].

(1.8) **Lemma.** *Let $C_1, \cdots, C_q$ and X be as above. Let L be a pseudo effective $\boldsymbol{Q}$-line bundle such that $(L-X)C_i \leqq 0$ for any $i=1, \cdots, q$. Then $L-X$ is pseudo effective.*

Proof. Let H be any semipositive $\boldsymbol{Q}$-line bundle. Since the matrix (C_iC_j) is non-singular, we have $Y \in V(C_1, \cdots, C_q)$ such that $YC_i = -HC_i$ for $i=1, \cdots, q$. $YC_i \leqq 0$ because H is semipositive. So Y is effective by (1.7). Hence $(L-X)C_i \leqq 0$ implies $(L-X)Y \leqq 0$. On the other hand, $(H+Y)C_i = 0$ implies that $(H+Y)X = 0$ and that $H+Y$ is semipositive by (1.4). So $(H+Y)L \geqq 0$. Combining these inequalities we obtain $(L-X)H = LH + XY \geqq XY - LY \geqq 0$. Q.E.D.

(1.9) **Lemma.** *Let $C_1, \cdots, C_q$ be prime divisors. Suppose that $I(C_1, \cdots, C_q)$ is negative semidefinite of type $(0, r)$ with $r<q$. Suppose in addition that $I(C_1, \cdots, C_r)$ is negative definite. Then, for each $j>r$, there is an effective $\boldsymbol{Q}$-divisor $X_j \in V(C_1, \cdots, C_r)$ such that $(C_j+X_j)C_i = 0$ for any $i=1, \cdots, q$.*

For a proof, see Zariski [8, p. 589].

(1.10) **Lemma.** *Let $C_1, \cdots, C_q$ be prime divisors and let L be a pseudo effective $\boldsymbol{Q}$-line bundle such that $LC_i \leqq 0$ for any i and $LC_j < 0$ for any $j>r$. Suppose that $I(C_1, \cdots, C_r)$ is negative definite. Then so is $I(C_1, \cdots, C_q)$.*

Proof. Assume that $I(C_1, \cdots, C_q)$ is not negative semidefinite. Then we have an effective $X \in V(C_1, \cdots, C_q)$ such that $X^2>0$. Replacing X by a positive multiple if necessary, we may assume that X is a usual divisor. $X^2>0$ implies $\kappa(X)=2$ by Riemann Roch theorem. Therefore, moreover, we may assume that the rational map defined by the linear system $|X|$ is birational (see [1]). Write $X=H+F$ where F is the fixed component of $|X|$. Then H is effective and $H \in V(C_1, \cdots, C_q)$. Hence $LH \leqq 0$ by the assumption on L. On the other hand, H is semipositive since $|H|$ has no fixed component. So $LH \geqq 0$ since L is pseudo effective. Thus we have $LH=0$. This implies $H \in V(C_1, \cdots, C_r)$ by assumption on L. Moreover, $H^2>0$ since $|X|$ defines a birational map. This contradicts that $I(C_1, \cdots, C_r)$ is negative definite. Thus we prove that $I(C_1, \cdots, C_q)$ is negative semidefinite.

Assume that $I(C_1, \cdots, C_q)$ is not negative definite. Then by (1.9) we have an effective $X_j \in V(C_1, \cdots, C_q)$ with $(C_j+X_j)C_i=0$. By assumption on L we have $L(C_j+X_j)<0$. On the other hand, C_j+X_j is semipositive by (1.4). So the above inequality contradicts that L is pseudo effective. Q.E.D.

(1.11) **Corollary.** *Let L be a pseudo effective $\boldsymbol{Q}$-line bundle.*

Then there are only finitely many prime divisors $\{C_i\}$ *with* $LC_i<0$.

Proof. Let $C_1, \cdots, C_q$ be prime divisors with $LC_i<0$. Then $I(C_1, \cdots, C_q)$ is negative definite by (1.10). So the Chern classes of C_i are linearly independent. Hence $q \leqq \rho =$ the Picard number of S. So there are at most ρ snch divisors.

(1.12) **Theorem.** *Let* L *be a pseudo effective* $\boldsymbol{Q}$*-line bundle. Then there is an effective* $\boldsymbol{Q}$*-divisor* N *such that*

a) $H=L-N$ *is semipositive,*

b) $HC_i=0$ *for any prime component* $C_1, \cdots, C_q$ *of* N,

c) $I(C_1, \cdots, C_q)$ *is negative definite.*

Moreover, N *is determined uniquely by the above properties.*

Proof. Let $C_1, \cdots, C_{q_1}$ be all the prime divisors such that $LC_i<0$ (cf. (1.11)). Take $N_1 \in V(C_1, \cdots, C_{q_1})$ such that $LC_i=N_1C_i$ for $i=1, \cdots, q_1$. Then N_1 is effective by (1.7) and $L_1=L-N_1$ is pseudo effective by (1.8). If L_1 is semipositive, then N_1 satisfies the desired condition. If not, let $C_{q_1+1}, \cdots, C_{q_2}$ be all the prime divisors with $L_1C_j<0$. Then $I(C_1, \cdots, C_{q_2})$is negative definite by (1.10). So we have $N_2 \in V(C_1, \cdots, C_{q_2})$ such that $L_1C_i=N_2C_i$ for $i=1, \cdots, q_2$. N_2 is effective by (1.7). $L_2=L_1-N_2$ is pseudo effective by (1.8). If L_2 is semipositive, then $N=N_1+N_2$ satisfies the desired condition. If not, we construct similarly $N_3 \in V(C_1, \cdots, C_{q_3})$ and $L_3=L_2-N_3$. Suppose that this process does not end till L_k. Then $I(C_1, \cdots, C_{q_k})$ is negative definite and $k \leqq q_k \leqq$the Picard number of S. Hence k cannot go to ∞. Thus we obtain a semipositive L_k after finite steps. Then $N=N_1+\cdots+N_k$ satisfies the desired condition.

The uniqueness of such N is proved by the same argument as in [8]. Q.E.D.

Remark. Zariski showed this result in case L is an effective $\boldsymbol{Q}$-divisor.

(1.13) The above N is called the (*arithmetically*) *negative part* of L and $L=H+N$ is called the *Zariski decomposition* of L. Note that H is pseudo effective by (1.8).

§ 2. Miyanishi's theory and the Zariski problem.

(2.1) A surface S is called *cylinderlike* if $S \cong A^1 \times C$ for a curve C. Using this notion, Miyanishi showed the following facts.

(2.2) **Theorem.** *Let* S *be a surface such that* $S \times A^1 \cong A^3$. *Suppose that* S *contains a cylinderlike open subset. Then* $S \cong A^2$ (cf. [5]).

(2.3) **Theorem.** *Let* $\bar{S}$ *be a complete surface and let* D *be an effective divisor on* $\bar{S}$ *with singularities at most normal crossings such that* $S=\bar{S}-D$ *is affine. Suppose that for any* $F \in \mathrm{Pic}(\bar{S})$ *we have* $|F+t(K+D)|=\varnothing$ *for* $t \gg 0$, *where* K *denotes the canonical bundle of* $\bar{S}$. *Then* S *contains a cylinderlike open subset* (cf. [6]).

(2.4) We remark that $\bar{\kappa}(S)=-\infty$ if $S\times A^1\cong A^3$ (see [3]). Using these results, we reduce the Zariski problem to the following

(2.5) **Proposition.** *Let S be a complete rational surface and let D be an effective divisor on it. Suppose that $\kappa(K+D)=-\infty$ where K is the canonical bundle of S. Then, for any $F\in \mathrm{Pic}(S)$, $\kappa(F+t(K+D))=-\infty$ for $t\gg 0$.*

Before proving this, we show a couple of lemmas.

(2.6) **Lemma.** *Let S be a complete surface and let $F, L\in \mathrm{Pic}(S)$. Suppose that $\kappa(F+t_jL)\geqq 0$ for a sequence $\{t_j\}$ with $\lim_{j\to\infty} t_j=\infty$. Then L is pseudo effective.*

Proof. Let H be any semipositive $\boldsymbol{Q}$-line bundle. Then $(F+t_jL)H\geqq 0$ since $|m(F+t_jL)|\neq\emptyset$ for some $m>0$. Letting $j\to\infty$, we infer that $LH\geqq 0$. Q.E.D.

(2.7) **Lemma.** *If both L and $-L$ is pseudo effective, then L is numerically equivalent to zero.*

Proof is easy.

(2.8) Now we prove (2.5). By (2.6), it suffices to show that $\kappa(K+D)\geqq 0$ if $K+D$ is pseudo effective. So let $K+D=H+N$ be the Zariski decomposition of $K+D$. If mH is a usual line bundle for a positive integer m, then $\kappa(mH)$ is defined and is independent of (cf. [1]). We denote it by $\kappa(H)$. Then $\kappa(K+D)\geqq\kappa(H)$ since N is effective. We have $H^2=(H+N)H=(K+D)H\geqq 0$ since $K+D$ is pseudo effective. If $H^2>0$, then we infer easily $\kappa(H)=2$ by the Riemann-Roch theorem. So it suffices to consider the case in which the equality holds. Then $KH=-DH\leqq 0$. This implies $h^0(tmH)+h^0(K-tmH)>0$ for any positive integer t by the Riemann-Roch theorem. Assume $\kappa(H)=-\infty$. Then $|K-tmH|\neq\emptyset$ for any $t>0$, and $-H$ is pseudo effective by (2.6). So H is numerically equivalent to zero by (2.7), and $H=0$ since S is rational. This contradicts $\kappa(H)=-\infty$. Q.E.D.

§3. **Topological characterization of A^2.** In this section everything is defined over $\boldsymbol{C}$. Details and proofs will be published elsewhere.

Combining Miyanishi's theory with (2.5), we can prove the following

(3.1) **Theorem.** *Let S be an affine smooth surface with $\bar{\kappa}(S)=-\infty$. Then S contains a cylinderlike open subset.*

This result enables us to study surfaces from a topological viewpoint. In particular we obtain

(3.2) **Theorem.** *Let S be an affine smooth surface with $\bar{\kappa}(S)=-\infty$, $H_1(S;\boldsymbol{Z})=H_2(S;\boldsymbol{Z})=0$. Then $S\cong A^2$.*

(3.3) **Corollary.** *Let S be an affine surface. Suppose that $S\times V\cong A^2\times V$ for some algebraic variety V. Then $S\cong A^2$.*

Because both $\bar{\kappa}$ and $H.(\ ;\boldsymbol{Z})$ are cancellation invariants.

(3.4) Using the result of Kambayashi [4], one can generalize (3.3) in the case of any ground field of characteristic zero.

(3.5) There is an affine smooth surface which is topologically contractible but not isomorphic to A^2 (see [7, § 3]). In case of this Ramanujan surface, we have $\bar{\kappa}=2$(see [2a]).

(3.6) It is desirable to establish the ruling theorem (3.1) also for non-affine surfaces. If this is done, then we can substitute $H_3(S; Z)=0$ for the condition "S is affine" in (3.2), (3.3).

(3.7) Finally we would like to ask the following

Question. Let M be a topologically contractible manifold with $\bar{\kappa}(M)=-\infty$. Then $M \cong A^n$?

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