24. *Experiments Concerning the Distribution of Squarefree Numbers*

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(Communicated by Kunihiko KODAIRA, M. J. A., March 12, 1979)

Let $Q(x)$ denote the number of squarefree integers not exceeding x. In this note some numerical results concerning $Q(x)$ obtained by the author will be reported. Before listing the results, we shall briefly refer to the theoretical property of $Q(x)$. Put for brevity

$$R(x)=Q(x)-\frac{6}{\pi^2}x.$$

As is well known, it can elementarily be proved that

$$R(x)=O(\sqrt{x}).$$

(cf. [1, p. 269]; [2, p. 582]; [3, p. 198]) Also, using the prime number theorem, or the fact that the Riemann zeta function $\zeta(s)$ has no zeros on the line $\sigma=1$, we can prove that

$$R(x)=o(\sqrt{x}).$$

(cf. [2, §162, p. 606])

On the other hand, by similar way as in [2], Fünftes Buch, Zwanzigster Teil, we can prove that

$$\liminf_{x\to\infty} x^{-1/4}R(x)<0, \qquad \limsup_{x\to\infty} x^{-1/4}R(x)>0,$$

so that $R(x)$ changes its sign infinitely often as x tends to infinity.

Here we list some results selected from the large amount of computer output.

The first line of Table I means that approximately $R(100)=.2$, $R(200)=.4$, $R(300)=.6$, $R(400)=-.1$, $R(500)=2.0$, $\cdots$. We omitted the figure below the first place of decimals for each $R(x)$.

The formula

$$Q(x)=\sum_{n\leq\sqrt{x}}\mu(n)\left[\frac{x}{n^2}\right]$$

was used. (cf. [1, p. 269]; [2, p. 581]) The computation was carried out at the Computer Center of Gakushuin University.

As is seen from the tables, the value of $R(x)$ frequently changes its sign. This phenomenon is in conformance with the above-mentioned theoretical result. Also it would be worth while noting that the absolute value of $R(x)$ is astonishingly small compared with x.

Table I. $R(x)$, $x=10^2(10^2)10^4$

.2	.4	.6	−.1	2.0	1.2	2.4	2.6	−.1	.0
−1.7	.4	1.6	2.9	3.1	4.3	1.5	1.7	−2.0	−.8
−.6	−.4	−.2	.9	3.1	1.3	.5	−.1	−2.9	.2
2.4	1.6	1.8	4.0	4.2	3.4	1.6	−1.1	.0	1.2
1.4	1.7	1.9	2.1	.3	.5	.7	2.9	4.1	2.3
−1.4	.7	−.0	−.8	−1.5	−3.3	−2.1	−1.9	−1.7	−1.5
.6	1.8	1.0	.2	.4	1.6	1.8	2.0	−.6	−.4
−.2	−1.0	−.8	−.6	1.5	−.2	1.9	2.1	4.3	1.5
2.7	1.9	3.2	3.4	1.6	2.8	3.0	5.2	6.4	1.6
2.8	4.0	5.2	4.4	4.6	2.8	4.1	6.3	4.5	3.7

$Q(10^4)=6083$

Table II. $R(x)$, $x=10^4(10^4)10^6$

3.7	1.4	4.1	−7.0	4.6	−2.6	1.1	−7.1	−.4	1.2
8.0	−3.2	8.4	2.2	.9	−1.3	2.3	−3.8	−2.1	−4.4
−1.6	.0	.7	3.4	−2.7	1.9	−2.3	.4	7.1	−.1
7.5	10.3	7.0	16.7	18.5	12.2	4.9	−3.2	.4	−5.8
−7.1	−15.3	−12.6	−11.9	−9.1	−6.4	−.7	−2.0	−6.2	−5.5
−2.8	−4.0	3.6	7.3	−1.9	3.8	3.5	2.2	−5.9	−2.2
2.4	11.1	8.9	12.6	9.3	12.1	7.8	11.5	6.2	9.0
6.7	13.4	7.2	−.0	−3.3	−2.5	−4.8	−2.1	2.5	.3
.0	−1.2	1.5	−2.7	−2.0	−.3	−.5	−3.8	−2.1	−3.3
.3	4.0	4.7	13.5	4.2	−.0	2.7	.4	−.8	−1.1

$Q(10^6)=607926$

Table III. $R(x)$, $x=10^6(10^6)10^8$

−1.1	22.7	−8.3	27.5	−2.5	−5.6	14.2	−13.8	−1.9	19.9
19.8	14.7	14.6	−7.4	−17.5	−37.6	−21.7	−36.8	−2.9	32.9
33.8	22.7	−.3	25.5	2.4	4.3	6.2	14.1	8.0	15.9
5.8	1.7	−1.3	21.5	6.4	−9.6	−22.7	−17.8	−16.9	−31.0
−23.1	−15.2	7.6	1.5	−.5	−12.6	−10.7	6.1	−4.9	−11.0

Table III (Continued)

−5.1	−10.2	−11.3	13.5	16.3	32.2	19.1	11.0	6.9	18.8
−13.2	19.6	19.5	15.4	2.3	12.2	15.1	32.0	11.9	4.8
52.7	59.6	60.5	31.4	27.3	22.2	35.1	−7.9	−5.0	−14.1
−27.2	−7.3	6.5	−24.5	9.3	−2.7	6.1	−12.9	−21.0	−25.1
−17.2	−28.3	−23.4	−30.5	−22.6	−24.7	−20.8	−48.9	−39.0	−16.1

$Q(10^8)=60792694$

Table IV. $R(x)$, $x=10^8(10^8)10^{10}$

−16.1	5.6	−4.5	64.2	−24.9	−19.1	−8.2	52.5	18.3	22.1
−9.0	−39.2	32.5	−3.5	4.2	4.0	−17.1	−42.3	−20.5	12.2
84.1	67.9	59.7	2.5	−29.6	77.1	.9	48.8	−5.3	66.4
60.2	17.0	−104.1	−75.3	−170.4	−182.6	−156.8	−127.0	32.7	−14.4
−37.6	−53.7	62.0	73.8	75.6	69.4	123.2	181.1	124.9	59.7
87.5	55.3	77.1	21.9	−13.1	20.6	21.4	−18.7	−7.9	11.8
75.6	112.5	99.3	4.1	−38.0	−73.2	19.5	−64.6	−52.7	−81.9
−13.1	−53.3	−30.5	−69.7	−86.9	−72.0	−22.2	36.5	57.3	2.1
−22.0	−22.2	40.6	−21.5	−.7	27.0	−26.1	−19.3	−18.5	11.3
−76.8	−126.0	−63.2	13.5	21.3	56.2	42.0	13.8	10.6	−76.5

$Q(10^{10})=6079270942$

Table V. $R(x)$, $x=10^{10}(10^{10})10^{12}$

−76.5	−182.0	141.3	18.8	−85.7	−28.2	33.2	27.6	251.1	94.5
−75.9	−366.4	−124.0	31.4	−89.1	−76.6	209.8	296.2	36.7	136.1
48.6	−124.8	−92.4	−330.9	−121.5	−50.0	−187.5	92.8	184.3	48.7
−286.7	−163.2	80.1	24.6	−30.9	−247.4	−131.9	−502.5	−304.0	−76.6
231.8	263.3	375.7	295.2	325.6	231.1	24.6	240.0	292.5	278.9
149.4	105.9	−74.6	14.8	221.2	152.7	−63.7	8.6	−145.8	98.5
−22.9	−86.4	99.9	26.4	−145.1	−56.6	−23.1	−230.7	−229.2	−208.8
−66.3	−358.8	−191.4	−513.9	−617.5	−604.0	−437.6	−477.1	−320.6	−299.2
−447.7	−191.3	−10.8	−115.3	−116.9	5.5	91.9	208.4	300.9	154.3
259.8	147.2	115.7	−110.7	−158.3	445.1	472.5	535.0	283.5	419.9

$Q(10^{12})=607927102274$

Table VI. $R(x)$, $x=10^{12}(10^{12})10^{14}$

419.9	142.9	630.9	−105.1	−314.1	778.8	−86.1	−262.2	−350.2	−258.2
−192.2	−230.3	757.6	−163.3	−136.3	−259.4	722.5	207.5	954.4	−11.5
−566.5	−540.5	−179.6	−1049.6	−413.6	−407.6	−909.7	−270.7	−159.7	−393.7
−419.8	−172.8	310.1	969.1	122.0	479.0	586.0	1395.9	1980.9	824.9
423.9	739.8	686.8	293.8	483.8	916.7	381.7	−466.2	−919.3	−1256.3
−417.3	−1428.3	−1482.4	−1170.4	−381.4	−694.4	−117.5	−528.5	−1008.5	−152.5
−128.6	−426.6	477.3	1023.3	244.2	410.2	421.2	659.1	942.1	−456.8
−284.8	194.0	−576.9	−376.9	−453.9	−260.0	−484.0	185.9	−42.0	−416.1
−479.1	−192.1	−709.2	87.7	42.7	898.7	−307.3	162.6	327.6	574.6
−107.4	384.5	1603.5	1088.5	497.4	738.4	544.4	4.3	35.3	544.3

$Q(10^{14})=60792710185947$

Table VII. $R(x)=10^{14}(10^{14})2\cdot 10^{15}$

544.3	1106.6	−259.9	−394.6	1046.7	991.1	723.4	−797.1	−45.8	108.4
−172.1	1737.1	4137.5	4208.8	1219.1	−977.5	−2309.1	−1010.8	−1715.5	−306.1

$Q(10^{15})=607927101854135$

$Q(2\cdot 10^{15})=1215854203707747$

References

[1] G. H. Hardy and E. M. Wright: An Introduction to the Theory of Numbers. 4th ed., Clarendon Press, Oxford (1975).
[2] E. Landau: Handbuch der Lehre von der Verteilung der Primzahlen. Leipzig, Teubner, reprinted by Chelsea (1953).
[3] A. Walfisz: Weylsche Exponentialsummen in der neueren Zahlentheorie. Berlin, VEB (1963).