16. *Generalized Multiple Wiener Integrals*

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§ 1. **Introduction.** The analysis of nonlinear functionals of Brownian motion $\{B(t)\}$, which we call simply *Brownian functionals*, can be expressed in terms of white noise $\{\dot{B}(t)\}$, $\dot{B}(t)=dB(t)/dt$. We are specifically interested in the so-called *causal calculus* where the propagation of time is taken into account. Intuitively speaking, $\{\dot{B}(t)\}$ may be taken to be a coordinate system of the basic space on which Brownian functionals are defined. At the same time, $\{\dot{B}(t)\}$ could be thought of as a system of variables of Brownian functionals. With this system the passage of time, say by h, can be represented explicitly as $\dot{B}(t)\to\dot{B}(t+h)$. In order to carry out the causal calculus we have naturally been led to the concept of generalized Brownian functionals ([3]). There we were inspired by P. Lévy's work [1] on functional analysis.

The purpose of this note is to discuss those generalized Brownian functionals by expressing them as generalized multiple Wiener integrals with respect to the generalized random measures formed from polynomials in the $\dot{B}(t)$'s. There we can see that our expression of generalized Brownian functionals is most fitting for the causal calculus in question.

§ 2. **Known results.** Brownian functionals with finite variance can be expressed in terms of white noise and realized as members of $(L^2)=L^2(\mathcal{S}^*,\mu)$, where $\mathcal{S}^*$ is the dual space of the Schwartz space $\mathcal{S}$ on R and μ is the probability distribution on $\mathcal{S}^*$ of the white noise $\{\dot{B}(t)\,;\,t\in R\}$ having the characteristic functional $C(\xi)$:

(1) $\quad C(\xi)=\exp[-\|\xi\|^2/2], \qquad \xi\in\mathcal{S},\ \|\cdot\|$ the $L^2(R)$-norm.

The Hilbert space (L^2) admits the Wiener-Itô decomposition

$$(L^2)=\sum_{n=0}^{\infty}\oplus\,\mathcal{H}_n, \tag{2}$$

where $\mathcal{H}_n$ is the *multiple Wiener integral* of degree n.

To visualize those members in (L^2) we have introduced the transformation $\mathcal{T}$ ([2]):

$$(\mathcal{T}\varphi)(\xi)=\int\exp[i\langle x,\xi\rangle]\varphi(x)d\mu(x), \qquad \varphi\in(L^2), \tag{3}$$

where $\langle\ ,\ \rangle$ stands for the canonical bilinear form that connects $\mathcal{S}$ and $\mathcal{S}^*$. The collection $\mathcal{F}=\{\mathcal{T}\varphi\,;\,\varphi\in(L^2)\}$ can be topologized so as to be isomorphic to (L^2) under the transformation $\mathcal{T}$. Indeed, with this topology $\mathcal{F}$ turns out to be the reproducing kernel Hilbert space with reproducing kernel $C(\xi-\eta)$, $(\xi,\eta)\in\mathcal{S}\times\mathcal{S}$.

Now set $\mathcal{T}(\mathcal{H}_n)=\mathcal{F}_n$. Then the decomposition (3) turns into that of $\mathcal{F}$:

$$(2)' \qquad \mathcal{F}=\sum_{n=0}^{\infty}\oplus\mathcal{F}_n.$$

The following theorem will play the key role in generalizing the concept of Brownian functionals.

Theorem 1. (i) *For $\varphi(x)\in\mathcal{H}_n$ we have the integral representation*

$$(4) \qquad (\mathcal{T}\varphi)(\xi)=i^nC(\xi)\int_{R^n}\cdots\int F(u_1,\cdots,u_n)\xi(u_1)\cdots\xi(u_n)du_1\cdots du_n,$$

where $F\in\widehat{L^2}(R^n)$ the class of symmetric $L^2(R^n)$-functions, and the map

$$\varphi\to F\in\widehat{L^2}(R^n), \qquad \varphi\in\mathcal{H}_n,$$

is one-to-one.

(ii) *Under the relationship established in* (i) *we have*

$$(5) \qquad \|\varphi\|_{(L^2)}=\sqrt{n!}\,\|F\|_{L^2(R^n)}.$$

Thus we may say that $\mathcal{H}_n$ (and hence $\mathcal{F}_n$) is isomorphic to the Hilbert space $\widehat{L^2}(R^n)$:

$$(6) \qquad \mathcal{H}_n\cong\mathcal{F}_n\cong\widehat{L^2}(R^n).$$

§ 3. Generalizes Brownian functionals. As was explained in § 1, $\{\dot{B}(t)\}$ may formally be thought of as a system of variables of Brownian functionals. It therefore seems to be reasonable to start with polynomials in the $\dot{B}(t)$'s. There we must meet again formal expressions that are hard to be interpreted. However, as soon as we come to the Hilbert space $\mathcal{F}$ we can immediately find out a way of extending $\mathcal{F}$ to a wider space which involves those functionals of ξ corresponding to polynomials in the $\dot{B}(t)$'s.

Observe the expression (4) and take F to be a member of the symmetric Sobolev space $H^{-(n+1)/2}(R^n)$ over R^n of order $-(n+1)/2$. Then, for example, we are given such a functional that

$$(7) \qquad i^nC(\xi)\int_{R^k}\cdots\int F(u_1,\cdots,u_k)\xi(u_1)^{n_1}\cdots\xi(u_k)^{n_k}du_1\cdots du_k, \quad \sum n_j=n,$$

which is an ordinary functional of ξ, while the corresponding Brownian functional must be a generalized one which is an integral of polynomials in $B(u_j)$'s of degree n.

Such an observation leads us to establish the following diagram (cf. (6)):

$$(8) \qquad \begin{array}{ccccc} \mathcal{H}_n^{(n)} & \subsetneq & \mathcal{H}_n & \subsetneq & \mathcal{H}_n^{(-n)} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \mathcal{F}_n^{(n)} & \subsetneq & \mathcal{F}_n & \subsetneq & \mathcal{F}_n^{(-n)} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \widehat{H}^{(n+1)/2}(R^n) & \subsetneq & \widehat{L^2}(R^n) & \subsetneq & \widehat{H}^{-(n+1)/2}(R^n), \end{array} \quad \begin{array}{l} \text{under } \mathcal{T} \\ \\ \text{via (7)} \end{array}$$

where the vertical double-headed arrows denote isomorphisms. The functional given by (7) is found in $\mathcal{F}_n^{(-n)}$ and it determines an $\mathcal{H}_n^{(-n)}$-functional.

§4. **Generalized random measures.** The integral in the formula (7) is understood to be an integral over R^n with respect not to the Lebesgue measure but to the product $\prod_k \delta^{n_k}(du_k^{n_k})$, where $\delta^m(du^m)$ is such a measure that

$$\int_{R^m}\cdots\int f(u_1,\cdots,u_m)\delta^m(du^m)=\int_{R^1} f(u,\cdots,u)du.$$

While the next formula for a functional given by the Hermite polynomial with parameter

$$\begin{aligned}&\mathcal{T}(H_m(\langle x,\chi_\Delta\rangle/\Delta u;1/\Delta u))(\xi)\\&\quad=\frac{i^m}{n!}C(\xi)\int_{R^n}\cdots\int \chi\Delta^m(u_1,\cdots,u_m)\xi(u_1)\cdots\xi(u_m)du^m\end{aligned}$$

(χ_A denotes the indicator function of A), $\langle x,\chi_\Delta\rangle$ being a version of $\Delta B(u)$, tells us that $\delta^m(du^m)$ determines a random measure, denote it by $H_m(\dot{B}(u);1/du)du$, which is the limit of $H_m(\langle x,\chi_\Delta\rangle/\Delta u;1/\Delta u)\Delta u$ in $\mathcal{H}_m^{(-m)}$ as $\Delta u\to 0$. We are now ready to define

$$M_n(dt)=H_n(\dot{B}(t);1/dt)dt \tag{9}$$

and also

$$\prod_{j=1}^k \cdot M_{n_j}(dt_j)\equiv M_{n_1}(dt_1)\cdot\cdots\cdot M_{n_k}(dt_k)=\begin{cases}\prod_{j=1}^k M_{n_j}(dt_j), & \text{if } t_1,\cdots,t_k \text{ are all distinct},\\ 0, & \text{otherwise}.\end{cases} \tag{10}$$

These are of course formal expressions, however they can be defined rigorously as in [3]. In fact, we can prove that the product $\prod_{j=1}^n\cdot M_{n_j}(dt_j)$ is a random measure in a generalized sense. We now have

Theorem 2. (i) *For $F\in\widehat{H^{(k+1)/2}}(R^k)$ the integral*

$$\varphi=\int_{R^k}\cdots\int F(u_1,\cdots,u_k)M_{n_1}(du_1)\cdot\cdots\cdot M_{n_k}(du_k),\qquad \sum_j n_j=n, \tag{11}$$

is defined, and the integral φ belongs to $\mathcal{H}_n^{(-n)}$. In addition,

$$(\mathcal{T}\varphi)(\xi)=\frac{i^n}{\prod_j n_j!}C(\xi)\int_{R^k}\cdots\int F(u_1,\cdots,u_k)\xi(u_1)^{n_1}\cdots\xi(u_k)^{n_k}du_1\cdots du_k \tag{12}$$

holds.

In view of this theorem the product (10) is called a *generalized random measure* of degree n, and the integral of the form (11) is said to be a *generalized multiple Wiener integral.* After P. Lévy a linear combination of generalized multiple Wiener integrals of several degrees is called a *normal Brownian functional.* The collection of normal Brownian functionals is an important class in the space of all generalized Brownian functionals in the causal calculus as is seen in [4] as well as in the forthcoming papers by the author.

At the moment, we show that multiplication of generalized random measures can be defined only for special cases. Multiplication by $M_1(dt)$ is possible as:

$$M_n(dt)\cdot M_1(ds)=M_n(dt)\bullet M_1(ds)+\delta_{t-s}M_{n-1}(dt)$$

(13) $$\left(\prod_{j=1}^{k}\bullet M_{n_j}(dt_j)\right)\cdot M_1(ds)=\sum_{j=1}^{k}\left\{M_{n_j}(dt)\cdot M_1(ds)\prod_{i\neq j}\bullet M_{n_i}(dt_i)\right\}.$$

For example

$$\int f(u)M_n(du)\cdot\int g(v)M_1(dv)$$
$$=\iint (f\otimes g)(u,v)M_n(du)\bullet M_1(dv)+\int f(u)g(u)M_{n-1}(du),$$

where $\otimes$ denotes the tensor product.

References

[1] P. Lévy: Problèmes concrets d'analyse fonctionnelle. Gauthier-Villars, Paris (1951).

[2] T. Hida: Stationary stochastic processes. Princeton Univ. Press, Princeton (1970).

[3] ——: Analysis of Brownian functionals. Carleton Univ. Lecture Notes, **13**, Ottawa (1975).

[4] ——: Analysis of Brownian Functionals. Math. Programming Study, **5**, 53–59 (1976).