14. *Structure of Logarithmic K3 Surfaces*

By Shigeru IITAKA
Department of Mathematics, Faculty of Science
University of Tokyo, Hongo, Tokyo 113

(Communicated by Kunihiko KODAIRA, M. J. A., Feb. 13, 1978)

1. By a surface, we shall mean a non-singular algebraic surface defined over $\boldsymbol{C}$. For complete surfaces, we have birational invariants such as geometric genus p_g, irregularity q and Kodaira dimension κ, by means of which the birational classification of surfaces has been discussed.

For open surfaces, in addition to those invariants, we have logarithmic geometric genus $\bar{p}_g$, logarihmic irregularity $\bar{q}$, and logarithmic Kodaira dimension $\bar{\kappa}$, which are proper birational invariants. For definition of them, see [2], [3], [5].

A $K3$ surface S is defined to be a complete surface S with $p_g(S)=1$ and $q(S)=\kappa(S)=0$. Moreover, if S is (relatively) minimal, the canoical divisor $K(S)\sim 0$ (which means that $K(S)$ is linearly equivalent to 0). Now, a *logarithmic* $K3$ *surface* S is defined to be a surface S with $\bar{p}_g(S)=1$ and $\bar{q}(S)=\bar{\kappa}(S)=0$. In this note we study the structure of logarithmic $K3$ surfaces. Details will appear elsewhere.

2. A pair $(\bar{S}, D)$ of a complete surface $\bar{S}$ and a divisor D with normal crossings is called a *∂-surface* and $S=\bar{S}-D$ is called the *interior* of $(\bar{S}, D)$. We say that $(\bar{S}, D)$ is *relatively ∂-mimimal* if $\bar{S}-D$ has no exceptional curves of the first kind and if D is minimal.

It is obvious that for a given surface S, there exists a ∂-surface $(\bar{S}, D)$ whose interior is S. $\bar{S}$ may be called a completion of S with ordinary boundary D.

3. Let S be a logarithmic $K3$ surface and let $(\bar{S}, D)$ be a ∂-surface whose interior is S. Then we have the following cases:

I) If $p_g(\bar{S})=1$, then $\bar{S}$ is a $K3$ surface. We put $D_A=0$ and $D_B=D$.

$\mathrm{II_a}$) If $p_g(\bar{S})=0$ and there is a component C_1 of D which is a non-singular elliptic curve, then $\bar{S}$ is a rational surface and the dual graph associated with D has no loops. We put $D_A=C_1$ and $D_A+D_B=D$.

$\mathrm{II_b}$) If $p_g(\bar{S})=0$ and each component of D is a rational curve, then $\bar{S}$ is a rational surface and the graph of D has one loop. Corresponding to the loop, we have a subboundary D_A which is circular (see § 4). We put $D=D_A+D_B$. In each case, we call S a logarithmic surface of type I) or $\mathrm{II_a}$) or $\mathrm{II_b}$).

4. Let $(\bar{S}, D)$ be a ∂-surface and $\sum_{j=1}^{r} C_j$ the irreducible decomposition of D. We say that D is a *circular boundary* if $r=1$ and C_1 is a rational curve with only one double point, or if $r=2$ and $(C_1, C_2)=2$, or if $r \geqq 3$ and each C_j is $\boldsymbol{P}^1$ satisfying that

$$(C_i, C_j)=1 \qquad \text{for } i-j \equiv \pm 1 \bmod r,$$
$$(C_i, C_j)=0 \qquad \text{for } i-j \not\equiv 0, \pm 1 \bmod r.$$

In general, the configuration of components of D defines a graph $\Gamma(D)$. The cyclotomic number of $\Gamma(D)$ is indicated by $h(\Gamma(D))$.

A divisor Y with normal crossings on a surface S is called a *curve of Dynkin type ADE* if each component $\mathcal{Y}_j$ of Y is a non-singular rational curve with $\mathcal{Y}_j^2=-2$ and if the graph $\Gamma(Y)$ corresponds to a direct sum of Dynkin diagrams A_n, D_n, E_l. In particular, $h(\Gamma(Y))=0$.

5. Let $(\bar{S}, D)$ be a ∂-surface whose interior S is a logarithmic $K3$ surface.

Theorem 1. *If $\bar{S}-D_A$ has no exceptional curves of the first kind, then $K(\bar{S})+D_A \sim 0$.*

Theorem 2. *If $(\bar{S}, D)$ is relatively ∂-minimal and if $\bar{S}-D_A$ has no exceptional curves of the first kind, then D_B is 0 or a curve of Dynkin type ADE. Furthermore, if S is of type II, then D_B is 0 or a curve of Dynkin type A.*

6. In general, let $(\bar{S}, D)$ be a ∂-surface and let $p \in D$. Consider a blowing up $\lambda: \bar{S}^1=Q_p(\bar{S}) \rightarrow \bar{S}$ with center p. Letting $D^1=\lambda^{-1}(D)$, we have

$$K(\bar{S}^1)+D^1=\lambda^*(K(\bar{S})+D)+(2-\nu)E$$

where $E=\lambda^{-1}(p)$ and ν is the multiplicity of D at p. Hence, if $\nu=2$, we call $\lambda: (\bar{S}^1, D^1) \rightarrow (\bar{S}, D)$ a *canonical blowing up*. We have

$$K(\bar{S}^1)+D^1=\lambda^*(K(\bar{S})+D).$$

If $\nu=1$, then defining D^* by $D^1=E+D^*$, we obtain

$$K(\bar{S}^1)+D^*=\lambda^*(K(\bar{S})+D).$$

We say that $(\bar{S}^1, D^*)$ (or $S^*=\bar{S}^1-D^*$) is a $1/2$-*point attachment* to $(\bar{S}, D)$ (or $S=\bar{S}-D$).

7. **Theorem 3.** *Let $(\bar{S}, D)$ be a relatively ∂-minimal ∂-surface such that D is a circular boundary. If $\bar{\kappa}(\bar{S}-D)=0$, then $(\bar{S}, D)$ is obtained from $(\boldsymbol{P}^2, H)$ by canonical blowing ups and downs and by attaching several $1/2$-points. Here H is a sum of three lines which have no common points.*

Theorem 4. *Let $(\bar{S}, C)$ be a relatively ∂-minimal ∂-surface such that C is an elliptic curve. If $\bar{\kappa}(\bar{S}-C)=q(\bar{S})=0$, then $(\bar{S}, C)$ is obtained from $(\boldsymbol{P}^2, E)$, E being a non-singular cubic curve, by canonical blowing ups and by several $1/2$-point attachments and detachments.*

8. We recall a result concerning quasi-abelian surfaces [6]. Let S be a surface with $\bar{q}(S)=2$ and $\bar{\kappa}(S)=0$. Such a surface is called a

logarithmic abelian surface. Let $(\bar{S}, D)$ be a ∂-surface whose interior is S. Then,

I) If $p_g(\bar{S})=1$, then $\bar{S}$ is birationally equivalent to an abelian surface. Put $D_A=0$ and $D_B=D$.

II) If $p_g(\bar{S})=0$ and $q(\bar{S})=1$, then $\bar{S}$ is a ruled surface of genus 1. Let $f: \bar{S} \rightarrow \Delta$ be the Albanese map. Then there is the horizontal component of D with respect to f, which is defined to be D_A. D_B is to satisfy $D=D_A+D_B$.

III) If $p_g(\bar{S})=q(\bar{S})=0$, then $\bar{S}$ is a rational surface and D consists of non-singular rational curves with $h(\Gamma(D))=1$. Define D_A to be a circular subboundary in D and D_B to be a divisor satisfying that $D=D_A+D_B$.

Theorem 5. *If $\bar{S}-D_A$ has no exceptional curves of the first kind, then $K(\bar{S})+D_A \sim 0$. Moreover, suppose that $D=D_A+D_B$ is minimal. Then $D=D_A$.*

Theorem 6. *In general, let S be an algebraic surface. Suppose that S is measure-hyperbolic. Then $\bar{\kappa}(S)=2$ or $\bar{\kappa}(S)=-\infty$ and $\bar{q}(S)=0$.*

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