78. *Remarks on the Differentiability of Solutions of Some Semilinear Parabolic Equations*

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1. Let H and V be a couple of real Hilbert spaces with $V \subset H \subset V^*$ algebraically and topologically. The norm and inner product of H are denoted by $|\ \ |$ and $(\ ,\ )$ respectively, and those of V are by $\|\ \ \|$ and $((\ ,\ ))$. Let $a(u, v)$ be a not necessarily symmetric bilinear form defined on $V \times V$ satisfying

$$|a(u, v)| \leqq C \|u\| \|v\|, \qquad a(u, u) \geqq \alpha \|u\|^2,$$

for some positive constants C and α. The associated linear operator is denoted by L:

$$a(u, v) = (Lu, v) \qquad u, v \in V.$$

Let ϕ be a properly convex lower semicontinuous convex function defined on V. Then the operator A defined by

$$Au = (Lu + \partial\phi(u)) \cap H$$

is a maximal monotone mapping on H to 2^H. For $u_0 \in \overline{D(A)}^H$ and $f \in W^{1,1}(0, T; H)$ let

$$u(t) = \lim_{n \to \infty} \prod_{i=1}^{n} \left\{1 + \frac{t}{n}\left(A - f\left(\frac{i}{n}t\right)\right)\right\}^{-1} u_0$$

be the solution of

$$du(t)/dt + Au(t) \ni f(t), \qquad u(0) = u_0$$

in the sense of M. G. Crandall–A. Pazy [4]. For this solution the following theorem holds. A related result is Theorem 3.2 of F. J. Massey, III [5], and in case L is symmetric also Corollary II. 2 of Chapter II of H. Brezis [2].

Theorem 1. *There exists a constant K such that*

$$|tD^+u(t)| \leqq K\left(|u_0 - v| + \int_0^t |f(s)|\, ds + t|A^0 v|\right) + \int_0^t |sf'(s) + f(s)|\, ds,$$

where v is an arbitrary element of $D(A)$.

Outline of the proof. It suffices to prove the theorem in the case $\min \phi = \phi(0) = 0$. First assume $u_0 \in D(A)$ and $f \in W^{1,2}(0, T; H)$. For $\varepsilon > 0$ let

$$\phi_\varepsilon(u) = \inf_v \left\{\frac{1}{2\varepsilon} \|u - v\|^2 + \phi(v)\right\}$$

be the Yosida approximation of ϕ, and A_ε be the operator defined by

$$A_\varepsilon u=(Lu+\partial\phi_\varepsilon(u))\cap H.$$

Let u_ε be the solution of the approximate equation

$$du_\varepsilon(t)/dt+A_\varepsilon u_\varepsilon(t)=f(t), \qquad u_\varepsilon(0)=u_{0\varepsilon}$$

where $u_{0\varepsilon}=(1+\varepsilon A_\varepsilon)^{-1}u_0$. It is not difficult to show that $u_\varepsilon\to u$ in $L^2(0,T;V)$. Hence it suffices to establish the corresponding estimate for u_ε with constants independent of ε, and we write u instead of u_ε to simplify the notation. Noting that $u(t)$ is Lipschitz continuous in $[0,T]$ it is easy to show that $u'\in L^2(0,T;V)$, and consequently $u''\in L^2(0,T;V^*)$. After a routine calculation we obtain

$$\begin{aligned}&\frac{1}{2}|tu'(t)|^2+\int_0^t(sLu'(s),su'(s))ds\\ &\qquad\leqq\int_0^t s|u'(s)|^2ds+\int_0^t(sf'(s),su'(s))ds,\end{aligned}\tag{1}$$

where we use the monotonicity of $\partial\phi_\varepsilon$. On the other hand noting

$$d\phi_\varepsilon(u(t))/dt=(\partial\phi_\varepsilon(u(t)),u'(t))$$

one easily deduce

$$\begin{aligned}&\int_0^t s|u'(s)|^2ds+\int_0^t(Lu(s),su'(s))ds+t\phi_\varepsilon(u(t))\\ &\qquad\leqq\int_0^t(f(s),su'(s))ds+\int_0^t\phi_\varepsilon(u(s))ds.\end{aligned}\tag{2}$$

Combining (1), (2) and the familiar inequality

$$\begin{aligned}&\frac{1}{2}|u(t)|^2+\int_0^t(Lu(s),u(s))ds+\int_0^t\phi_\varepsilon(u(s))ds\\ &\qquad\leqq\frac{1}{2}\left(|u_{0\varepsilon}|+\int_0^t|f(s)|ds\right)^2\end{aligned}$$

and using Lemma A.5 of [1] we can establish the desired estimate. The result in the general case is established by approximating u_0 and f in the obvious manner.

2. As an application we consider the following unilateral problem

$$\left.\begin{array}{ll}\partial u/\partial t+\mathcal{L}u\geqq f, & u\geqq\Psi\\ (\partial u/\partial t+\mathcal{L}u-f)(u-\Psi)=0\end{array}\right\}\quad\text{in }\Omega\times[0,T],$$

$$-\partial u/\partial n\in\beta(x,u)\quad\text{on }\Gamma\times[0,T],\ u(x,0)=u_0(x)\quad\text{in }\Omega,$$

where $\mathcal{L}$ is a linear elliptic operator of second order, and slightly improve the estimate in the previous paper [6].

Let Ω be a not necessarily bounded domain in R^N with smooth boundary Γ. Let

$$a(u,v)=\int_\Omega\left(\sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\sum_{i=1}^N b_i\frac{\partial u}{\partial x_i}v+cuv\right)dx$$

be a bilinear form defined on $H^1(\Omega)\times H^1(\Omega)$. The coefficients a_{ij}, b_i are bounded and continuous together with their first derivatives and c is bounded and measurable in Ω. The matrix $\{a_{ij}(x)\}$ is uniformly positive definite and there exists a positive constant α such that $c\geqq\alpha$,

$c-\sum_{i=1}^{N}\partial b_i/\partial x_i \geqq \alpha$ almost everywhere in Ω. We denote by $\mathcal{L}$ the differential operator associated with the bilinear form $a(u,v)$:

$$\mathcal{L}=-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)+\sum_{i=1}^{N}b_i\frac{\partial}{\partial x_i}+c.$$

Let $j(x,r)$ be a function defined on $\Gamma\times(-\infty,\infty)$ such that for each $x\in\Gamma$ $j(x,r)$ is a properly convex lower semicontinuous function of r and $j(x,r)\geqq j(x,0)=0$. We denote by $\beta(x,\cdot)=\partial j(x,\cdot)$ the subdifferential of $j(x,r)$ with respect to r. As for the regularity with respect to x we assume that for each $t\in(-\infty,\infty)$ and $\lambda>0$ $(1+\lambda\beta(x,\cdot))^{-1}(t)$ is a measurable function of x (cf. B. D. Calvert–C. P. Gupta [3]). Let $\psi: L^2(\Gamma)\to[0,\infty]$ be the convex function defined by

$$\psi(u)=\begin{cases}\int_\Gamma j(x,u(x))d\Gamma, & j(u)\in L^1(\Gamma)\\ \infty, & \text{otherwise.}\end{cases}$$

Unless $rj(x,r)=\infty$ as $r\neq 0$ (namely the boundary condition is of Dirichlet type), we assume that $\sum_{i=1}^{N} b_i\nu_i\geqq 0$ on Γ where $\nu=(\nu_1,\cdots,\nu_N)$ is the outer normal vector to Γ.

By $G(A)$ we denote the graph of the mapping A.

The operator $L_p: L^p(\Omega)\to L^p(\Omega)$, $1\leqq p<\infty$, is defined as follows:

(i) for $p=2$ $f\in L_2u$ if $u\in H^1(\Omega)$, $\psi(u|_\Gamma)<\infty$ and

$$a(u,v-u)+\psi(v|_\Gamma)-\psi(u|_\Gamma)\geqq\int_\Omega f(v-u)dx$$

for every $v\in H^1(\Omega)$ such that $\psi(v|_\Gamma)<\infty$;

(ii) for $p\neq 2$, $G(L_p)=$the closure of $G(L_2)\cap(L^p(\Omega)\times L^p(\Omega))$ in $L^p(\Omega)\times L^p(\Omega)$.

In what follows we assume $1<p<2<p^*=Np/(N-p)$. Let Ψ be a function such that $\Psi\in W^{2,p}(\Omega)\cap W^{1,1}(\Omega)$, $\mathcal{L}\Psi\in L^1(\Omega)$ and $\partial\Psi/\partial n+\beta^-(x,\Psi)\leqq 0$ on Γ, where

$\beta^-(x,r)=\min\{z: z\in\beta(x,r)\}$ if $r\in D(\beta(x,\cdot))$,

$\beta^-(x,r)=\infty$ if $r\notin D(\beta(x,\cdot))$ and $r\geqq\sup D(\beta(x,\cdot))$,

$\beta^-(x,r)=-\infty$ if $r\notin D(\beta(x,\cdot))$ and $r\leqq\inf D(\beta(x,\cdot))$.

We define the mapping M_p by

$D(M_p)=\{u\in L^p(\Omega): u\geqq\Psi \text{ a.e. in } \Omega\}$,

$M_pu=\{g\in L^p(\Omega): g\leqq 0 \text{ a.e.}, g(x)=0 \quad \text{if } u(x)>\Psi(x)\}$,

and similarly M_1 with $L^1(\Omega)$ in place of $L^p(\Omega)$.

The operator A_q, $1\leqq q\leqq p^*$, is defined as follows:

(i) $A_p=L_p+M_p$,

(ii) $A_1=L_1+M_1$,

(iii) for $1<q\leqq p^*$, $q\neq 2$, $G(A_q)=$the closure of $G(A_p)\cap(L^q(\Omega)\times L^q(\Omega))$ in $L^q(\Omega)\times L^q(\Omega)$.

Proposition. *A_q is m-accretive and*

$$\overline{D(A_q)}=\{u \in L^q(\Omega): u \geqq \Psi \text{ a.e. in } \Omega\}.$$

It is known that L_2+M_2 is not m-accretive in general under our hypothesis.

Let $L: H^1(\Omega) \to H^1(\Omega)^*$ be the operator associated with the bilinear form $a(u, v): a(u, v)=(Lu, v)$ for $u, v \in H^1(\Omega)$, and ϕ be the convex function on $H^1(\Omega)$ defined by

$$\phi(u)=\begin{cases} \int_\Gamma j(x, u(x))d\Gamma, & u \geqq \Psi \text{ a.e., and } j(u|_\Gamma) \in L^1(\Gamma), \\ \infty, & \text{otherwise.} \end{cases}$$

The effective domain $D(\phi)$ of ϕ is not empty since it follows that $\Psi^+ \in D(\phi)$ from the present hypothesis. Then it is not difficult to show that A_2 coincides with the operator defined by

$$Au=(Lu+\partial\phi(u)) \cap L^2(\Omega).$$

Thus applying Theorem 1 and a comparison theorem we obtain

Theorem 2. *Suppose that $\Psi \leqq u_0 \in L^q(\Omega)$ and $f \in W^{1,1}(0, T; L^q(\Omega) \cap L^r(\Omega))$, $1 \leqq q \leqq 2 \leqq r$. Then for the solution of*

$$du(t)/dt+A_q u(t) \ni f(t), \qquad 0<t \leqq T,\ u(0)=u_0,$$

we have

$$\|D^+u(t)\|_r \leqq C_0\Big\{t^{\beta-1}(\|\Psi\|_2+\|v\|_2+t\,\|A_2^\circ v\|_2) + t^{\gamma-1}\|u_0\|_q+t^\delta\,\|(L\Psi)^+\|_p+t^{\beta-1}\int_0^t \|f(s)\|_2\,ds + t^{\beta-1}\int_0^t s\,\|f'(s)\|_2\,ds+\int_0^t \|f'(s)\|_r\,ds\Big\}$$

where v is an arbitrary element of $D(A_2)$, $\beta=N(r^{-1}-2^{-1})/2$, $\gamma=N(r^{-1}-q^{-1})/2$, $\delta=N(r^{-1}-p^{-1})/2$ and $\|\ \ \|_r$ denotes the norm of $L^r(\Omega)$.

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