77. Studies on Holonomic Quantum Fields. X

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(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1978)

In this series of articles entitled "holonomic quantum fields", we intend to develop an exact model theory of quantum fields. Thus far we have expounded the subject in the simplest and most typical case of 2-dimensional space-time. Let us formulate here the mathematical situation in a form applicable to any space-time dimensions.

Consider a classical scattering problem caused by a given external field $A(x)$. The associated scattering operator $T=T[A]$ is then a rotation in the space W of free wave functions. The quantal scattering is attributed to the corresponding Clifford element $g \in G(W)$ (or $g\otimes g^{-1} \in G(W\oplus W)$), which we regard as the field operator of an "extended object" $A(x)$. Our aim is to show that the vacuum expectation value $\tau[T]$ of g, and further the operator g itself, is completely characterized and controlled in the language of classical mathematics.

In this and the coming notes we shall perform the above program in the case of 2-dimensional massless Dirac fields. The general framework is presented in X–§1. By specializing the formulas in X–§1 we derive variational formulas for $\log \tau[T]$ regarded as a functional of the external field A (X–§2) and also of the rotation T (XI–§1), both in an exact and closed manner. We note that in this context the Riemann-Hilbert problem arises more naturally as a 2-dimensional massless field theory, rather than the 1-dimensional formulation given in [1], [3]. Finally in XI–§2 we shall apply these results to give a non-abelian extension of the Szegö's theorem concerning the Toeplitz determinant.

Application of the scheme developed here to higher dimensional problems will be the theme of our subsequent publications.

1. Let W be an orthogonal vector space equipped with the inner product $\langle\ ,\ \rangle$. Given a rotation $T\in O(W)$, let g be an element of the Clifford group $G(W)$ which induces $T: T_g=T$ [2]. Then $g\otimes g^{-1} \in G(W\oplus W)$ does not depend on the choice of g and is uniquely determined by T. Assuming $\dim W=N$ to be even, we fix a holonomic decomposition $W=V^{\dagger}\oplus V$. The corresponding expectation values on $A(W)$ and on $A(W\oplus W)$ are denoted by $\langle\ \rangle_W$ and $\langle\ \rangle_{W\oplus W}$ respectively, so that $\langle a_1\otimes a_2\rangle_{W\oplus W}=\langle a_1\rangle_W\langle a_2\rangle_W (a_1, a_2 \in A(W))$. Consider now the decomposition of a rotation T and of T^{-1}

$$(1)\qquad T=Y_+^{-1}Y_-,\quad T^{-1}=Z_+^{-1}Z_-\qquad (Y_\pm, Z_\pm \in \mathrm{End}\,(W))$$

satisfying (cf. [2], p. 260)

$$(2)\qquad E_-Y_+^{\pm 1}E_+=0,\qquad E_+Y_-^{\pm 1}E_-=0$$
$$E_-Z_+^{\pm 1}E_+=0,\qquad E_+Z_-^{\pm 1}E_-=0.$$

Here E_+ (resp. E_-) denotes the projection operator onto $V^\dagger$ (resp. V). The choice of $Y_\pm, Z_\pm$ is arbitrary within the replacement $Y_\pm \mapsto CY_\pm$, $Z_\pm \mapsto C'Z_\pm$, C, C' being invertible and commutative with $E_\pm$. We set

$$(3)\qquad \Phi[T]=Y_-^{-1}E_-Y_+-Z_+^{-1}E_-Z_-$$
$$=-Y_-^{-1}E_+Y_++Z_+^{-1}E_+Z_-.$$

Next choose a basis $v_1, \cdots, v_N$ of W, and denote by $J=(\langle v_j, v_k\rangle)_{j,k=1,\dots,N}$, $K=(\langle v_jv_k\rangle)_{j,k=1,\dots,N}$ the tables of inner product and of expectation value, respectively. Note that $E_+=J^{-1}K$, $E_-=J^{-1}{}^tK$ in this basis. We put

$$(4)\qquad R[T]=(Y_+^{-1}-Y_-^{-1})(E_+Y_-+E_-Y_+)J^{-1}$$
$$R[T^{-1}]=(Z_+^{-1}-Z_-^{-1})(E_+Z_-+E_-Z_+)J^{-1}.$$

Clearly the definitions (3), (4) are independent of the choice of $Y_\pm, Z_\pm$.

Theorem 1. *We have*

$$(5)\qquad \langle g\otimes g^{-1}\rangle_{W\oplus W}=\tau[T]$$
$$\mathrm{Nr}\,(g\otimes g^{-1})=\tau[T]\exp(\rho[T]/2)$$

where $\tau[T]$, $\rho[T]$ are given by the following formulas:

$$(6)\qquad \tau[T]^2=\det(E_++E_-T)\det(E_++E_-T^{-1})$$
$$=\det(E_+Y_-^{-1}+E_-Y_+^{-1})Y_-\cdot\det(E_+Z_-^{-1}+E_-Z_+^{-1})Z_-$$
$$\rho[T]=\sum_{j,k=1}^N r_{jk}^{(1)}v_j^{(1)}v_k^{(1)}+\sum_{j,k=1}^N r_{jk}^{(2)}v_j^{(2)}v_k^{(2)}.$$

Here $(r_{jk}^{(1)})_{j,k=1,\dots,N}=R[T]$, $(r_{jk}^{(2)})_{j,k=1,\dots,N}=R[T^{-1}]$, and $v_j^{(1)}, v_j^{(2)}$ denote the first and second copies of v_j in $W\oplus W$ respectively $(j=1, \cdots, m)$.

The variation of $\tau[T]$ as a functional of T is given by

$$(7)\qquad 2\delta\log\tau[T]=\mathrm{trace}\,\delta T\cdot\Phi[T].$$

These results are equally valid in the symplectic framework, provided that we replace J, E_+, E_- by $H=K-{}^tK$, $E'_+=H^{-1}K$, $E'_-=-H^{-1}{}^tK$, respectively.

2. Now we shall discuss the massless theory in 2-dimensional Minkowski space-time $X^{Min}=\boldsymbol{R}^2$. We use the following notations: $x=(-x^-, x^+)$, $\pm x^\pm=(\pm x^0+x^1)/2$, $d^2x=d(-x^-)dx^+$, $p^\pm=p^0\pm p^1$, $p\cdot x=p^0x^0-p^1x^1=p^+x^-+p^-x^+$. Let W be the space of "wave functions" $w={}^t(w_1, \cdots, w_m)$, namely m-tuples of functions on X^{Min} satisfying the 2-dimensional Weyl equation (the massless version of Dirac equation)

$$(8)\qquad \frac{\partial}{\partial x^+}w_j(x)=0\qquad (j=1, \cdots, m).$$

We equip W with the inner product $\langle w, w'\rangle=\sum_{j=1}^N\int_{-\infty}^{+\infty}d(-x^-)w_j(x)w'_j(x)$ to make it an orthogonal space. Denote by $V^\dagger$ (resp. V) the subspace of W consisting of w's whose Fourier transforms $\hat{w}_j(p^+)2\pi\delta(p^-)$

$=\int d^2xw_j(x)e^{ip\cdot x}$ $(j=1, \cdots, m)$ are supported on $p^+\geqq 0$ (resp. $p^+\leqq 0$). We have then a holonomic decomposition $W=V^\dagger\oplus V$. As in II–§1 [3] we introduce an ideal element $\psi_k(x_0)$ of W, whose j-th component is given by $\delta_{jk}\delta(-x^-+x_0^-)$. The tables of inner product and expectation value in the basis $\{\psi_k(x_0)\}$ are the invariant delta functions defined by

$$(9)\qquad \begin{aligned} D(x-x')&=(\langle\psi_j(x), \psi_k(x')\rangle)_{j,k=1,\cdots,m}=\delta(-x^-+x'^-)\cdot I_m\\ &=D^{(+)}(x-x')+D^{(-)}(x-x')\\ D^{(+)}(x-x')&=(\langle\psi_j(x)\psi_k(x')\rangle)_{j,k=1,\cdots,m}=\frac{1}{2\pi}\frac{i}{-x^-+x'^-+i0}\cdot I_m\\ D^{(-)}(x-x')&=(\langle\psi_k(x')\psi_j(x)\rangle)_{j,k=1,\cdots,m}=\frac{1}{2\pi}\frac{-i}{-x^-+x'^--i0}\cdot I_m, \end{aligned}$$

where I_m is the $m\times m$ unit matrix. We shall also make use of the following Green's functions:

$$(10)\qquad \begin{aligned} D_c(x-x')&=(\langle \boldsymbol{T}(\psi_j(x)\psi_k(x'))\rangle)_{j,k=1,\cdots,m}\\ &=\theta(x^+-x'^+)D^{(+)}(x-x')-\theta(x'^+-x^+)D^{(-)}(x-x')\\ D_c^*(x-x')&=(\langle \boldsymbol{T}^*(\psi_j(x)\psi_k(x'))\rangle)_{j,k=1,\cdots,m}\\ &=\theta(x^+-x'^+)D^{(-)}(x-x')-\theta(x'^+-x^+)D^{(+)}(x-x')\\ D_{ret}(x-x')&=\theta(x^+-x'^+)D(x-x')\\ D_{adv}(x-x')&=-\theta(x'^+-x^+)D(x-x'). \end{aligned}$$

In (10) $\boldsymbol{T}$(resp. $\boldsymbol{T}^*$) signifies the time ordered (resp. anti-time ordered) product: $\boldsymbol{T}(\psi_{j_1}(x_1)\cdots\psi_{j_r}(x_r))=\sum\varepsilon_\sigma\theta(x^0_{\sigma(1)}-x^0_{\sigma(2)})\cdots\theta(x^0_{\sigma(r-1)}-x^0_{\sigma(r)})\psi_{j_{\sigma(1)}}(x_{\sigma(1)})\cdots\psi_{j_{\sigma(r)}}(x_{\sigma(r)})$, $\boldsymbol{T}^*(\psi_{j_1}(x_1)\cdots\psi_{j_r}(x_r))=\sum\varepsilon_\sigma\theta(-x^0_{\sigma(1)}+x^0_{\sigma(2)})\cdots\theta(-x^0_{\sigma(r-1)}+x^0_{\sigma(r)})\psi_{j_{\sigma(1)}}(x_{\sigma(1)})\cdots\psi_{j_{\sigma(r)}}(x_{\sigma(r)})$, where the sum extends over permutation σ of indices $\{1, \cdots, r\}$, and $\varepsilon_\sigma=$signature of σ. These functions (9), (10) are interrelated through $D_c=D_{ret}-D^{(-)}=D_{adv}+D^{(+)}$, $D_c^*=D_{ret}-D^{(+)}=D_{adv}+D^{(-)}$. Note that $D^{(\pm)}$ represents the kernel function of the projection operator $E_\pm$.

Consider now the Weyl equation with external source

$$(11)\qquad \left(\frac{\partial}{\partial x^+}-A(x)\right)w(x)=0.$$

Here $A(x)$ denotes a compactly supported, smooth $m\times m$ matrix valued function. We assume ${}^tA(x)=-A(x)$. (The general case is reduced to this case by virtue of the inclusion of Lie algebras $\mathfrak{gl}(m)\subset\mathfrak{o}(2m)$.) A solution $w(x)$ of (11) is in one-to-one correspondence with its asymptotic wave functions w_{in} or $w_{out}\in W$ defined by

$$(12)\qquad w_{\substack{in\\out}}(x)=w(x)-\int d^2x' D_{\substack{ret\\adv}}(x-x')A(x')w(x').$$

It is easy to verify that the scattering operator $T[A]: W\to W$, $w_{in}\mapsto w_{out}$, induces a rotation in W. Explicitly it is given by the multiplication operator

$$(13)\qquad (T[A]w_{in})(x)=M[A](-x^-)\cdot w_{in}(x)$$

$$M[A](-x^-)=\sum_{n=0}^{\infty}\int\cdots\int d^2x_1\cdots d^2x_n D(x-x_1)A(x_1)D_{ret}(x_1-x_2)A(x_2)$$
$$\cdots D_{ret}(x_{n-1}-x_n)A(x_n)$$
$$=1+\sum_{n=1}^{\infty}\int\cdots\int dx_1^+\cdots dx_n^+\theta(x_1^+-x_2^+)\cdots\theta(x_{n-1}^+-x_n^+)$$
$$\times A(-x^-,x_1^+)\cdots A(-x^-,x_n^+).$$

Note that ${}^tM[A]^{-1}=M[A]$.

Theorem 2. *The τ-function $\tau[T[A]]$ (which we denote by $\tau[A]$) corresponding to $\tau[A]$ is given by*

$$(14)\qquad 2\log\tau[A]=\mathrm{trace}\,(\log(1-D_cA)+\log(1-D_c^*A)$$
$$-\log(1-D_{ret}A)-\log(1-D_{adv}A))$$
$$=-\sum_{n=1}^{\infty}\frac{1}{n}\int d^2x\,\mathrm{trace}\,\Psi^{(n)}(x,x;A)$$

where

$$(15)\qquad \Psi^{(n)}(x,x';A)=\int\cdots\int d^2x_1\cdots d^2x_{n-1}$$
$$\{D_c(x-x_1)A(x_1)\cdots D_c(x_{n-1}-x')A(x')$$
$$+D_c^*(x-x_1)A(x_1)\cdots D_c^*(x_{n-1}-x')A(x')$$
$$-D_{ret}(x-x_1)A(x_1)\cdots D_{ret}(x_{n-1}-x')A(x')$$
$$-D_{adv}(x-x_1)A(x_1)\cdots D_{adv}(x_{n-1}-x')A(x')\}.$$

In (14) D_c, A, etc. are regarded as integral operators with kernels $D_c(x-x')$, $A(x)\delta(-x^-+x'^-)\delta(x^+-x'^+)$, etc. The crucial point is that $\Psi^{(n)}(x,x;A)$ is well defined, although individual terms in (15) are singular on the diagonal.

To obtain a closed expression for the variation $\delta\log\tau[A]$, we introduce several Green's functions $G=D_{ret}^A$, D_{adv}^A, D_c^A, D_c^{*A} for (11):

$$(16)\qquad \left(\frac{\partial}{\partial x^+}-A(x)\right)G(x,x')=\delta(-x^-+x'^-)\delta(x^+-x'^+)I_m$$

with the following characteristic boundary conditions.

$$(17)\qquad D_{\substack{ret\\adv}}^A(x,x')=0\qquad \text{for } x^+\lesseqgtr x'^+.$$

$$(18)\qquad D_c^A(x,x'),\quad D_c^{*A}(x,x')\to 0\qquad (|-x^-|\to\infty),$$
$$(D_c^A)_{x',in}\in V,\qquad (D_c^{*A})_{x',in}\in V^\dagger$$
$$(D_c^A)_{x',out}\in V^\dagger,\qquad (D_c^{*A})_{x',out}\in V.$$

Here, for fixed x', $(D_c^A)_{x',in}$ and $(D_c^{*A})_{x',in}$ (resp. $(D_c^A)_{x',out}$ and $(D_c^{*A})_{x',out}$) denote elements of W which coincide with $D_c^A(x,x')$ and $D_c^{*A}(x,x')$ in the region $x^+\ll x^{+\prime}$ (resp. $x^+\gg x'^+$), where the letters satisfy the free equation (8). We set

$$(19)\qquad \Psi(x,x';A)=D_c^A(x,x')+D_c^{*A}(x,x')$$
$$-D_{ret}^A(x,x')-D_{adv}^A(x,x').$$

Theorem 3. *The restriction $\Psi|_{x=x'}$ is well defined, and we have*

$$(20)\qquad 2\delta\log\tau[A]=-\int d^2x\,\mathrm{trace}\,\delta A(x)\cdot\Psi(x,x;A).$$

References

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