72. *Parallel Vector Fields and the Betti Number*

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Introduction. Let M^n be an n dimensional connected compact orientable smooth Riemannian manifold. In the previous paper [3] we showed that the Betti numbers of M^n with one or two parallel vector fields satisfy some inequalities. In this note we shall generalize these results to the case of M^n admitting r parallel vector fields $(1\leqq r \leqq n)$. A trivial example of such M^n is the Riemannian product $T^r\times M^{n-r}$, where T^r is the flat r-torus and M^{n-r} is any Riemannian manifold.

1. Preliminaries. Let $\mathcal{H}_p$ be the vector space of harmonic p-forms on M^n. $\dim \mathcal{H}_p$ is equal to the p-th Betti number b_p. We make a convention that $\mathcal{H}_p=\{0\}$ for $p>n$ or $p<0$ and hence all operators act trivially on such spaces. Throughout the paper we shall denote by p any integer.

Let u be a vector field on M^n. By the natural identification with respect to the Riemannian metric, u is identified with a 1-form which will be denoted by u again. $e(u)$ and $i(u)$ denote respectively the operators of exterior and interior product by u. For a p-form ω, we have $e(u)\omega=u\wedge\omega$ and

$$(i(u)\omega)(X_1, \cdots, X_{p-1})=\omega(u, X_1, \cdots, X_{p-1})$$

where $X_1, \cdots, X_{p-1}$ are tangent vectors. These operators satisfy $e(u)^2=i(u)^2=0$. $i(u)$ is an anti-derivation and hence

$$i(u)e(u)+e(u)i(u)=I \tag{1}$$

holds for a unit vector field u, where I is the identity on p-form.

2. Parallel vector fields. Let u be a parallel vector field on M^n. First we notice that $\omega\in\mathcal{H}_p$ implies $e(u)\omega\in\mathcal{H}_{p+1}$ and $i(u)\omega\in\mathcal{H}_{p-1}$.

Now we assume that M^n admits r $(1\leqq r\leqq n)$ linearly independent parallel vector fields $u_1, \cdots, u_r$. Making use of the Schmidt process, we may suppose that $u_1, \cdots, u_r$ are orthonormal, i.e.,

$$i(u_k)u_j=\delta_{kj} \qquad (1\leqq k,\ j\leqq r).$$

$a_1, \cdots, a_k$ $(1\leqq a_1, \cdots, a_k\leqq r)$ being integers, let us define

$$i_{a_1\cdots a_k}=i(u_{a_1})\cdots i(u_{a_k}) : \mathcal{H}_p\to\mathcal{H}_{p-k},$$
$$e_{a_1\cdots a_k}=e(u_{a_1})\cdots e(u_{a_k}) : \mathcal{H}_p\to\mathcal{H}_{p+k}.$$

Lemma. *For* $1\leqq s\leqq r$, *we have*

$$I=-\sum_{k=1}^{s}\sum_{1\leqq a_1<\cdots<a_k\leqq s}(-1)^{k(k+1)/2}e_{a_1\cdots a_k}i_{a_1\cdots a_k}+(-1)^{s(s-1)/2}i_{1\cdots s}e_{1\cdots s}. \tag{2}$$

Proof. When $s=1$, (2) is nothing but (1). Suppose that (2) is true for $s-1$. Taking account of (1) and $i(u_k)e(u_j)=-e(u_j)i(u_k)$ for $j\neq k$, we have

$$i_{1\cdots s}e_{1\cdots s}=(-1)^{s-1}(i_{1\cdots(s-1)}e_{1\cdots(s-1)})(i_s e_s)$$
$$=(-1)^{(s-1)(s-2)/2+(s-1)}\left\{I+\sum_{k=1}^{s-1}\sum_{1\leqq a_1<\cdots<a_k<s}(-1)^{k(k+1)}e_{a_1\cdots a_k}i_{a_1\cdots a_k}\right\}\times(I-e_s i_s),$$

from which we know (2) to be valid for s.

3. Theorems. Let $u_1,\cdots,u_r$ be orthonormal parallel vector fields on M^n. Putting $i(u)_p=i(u)$ for any integer p, we consider the linear mapping $i(u)_p:\mathcal{H}_p\to\mathcal{H}_{p-1}$ and define

$$\mathcal{K}_p(u_h)=\operatorname{Ker} i(u_h)_p,\quad k_p(u_h)=\dim\mathcal{K}_p(u_h),\quad 1\leqq h\leqq r,$$
$$\mathcal{K}_p^{(s)}=\mathcal{K}_p(u_1)\cap\cdots\cap\mathcal{K}_p(u_s),\quad k_p^{(s)}=\dim\mathcal{K}_p^{(s)},\quad 1\leqq s\leqq r,$$

where $\mathcal{K}_p^{(0)}=\mathcal{H}_p$ and $k_p^{(0)}=b_p$ by convention. Then $i(u_s)_p|\mathcal{K}_p^{(s-1)}:\mathcal{K}_p^{(s-1)}\to\mathcal{H}_{p-1}$ has the image in $\mathcal{K}_{p-1}^{(s)}$.

Theorem 3.1. *If M^n admits r orthonormal parallel vector fields $u_1,\cdots,u_r$, then we have*

$$\operatorname{Ker}(i(u_s)_p|\mathcal{K}_p^{(s-1)})=\mathcal{K}_p^{(s)},\qquad \operatorname{Im}(i(u_s)_p|\mathcal{K}_p^{(s-1)})=\mathcal{K}_{p-1}^{(s)},$$

and hence

$$\mathcal{K}_p^{(s-1)}\cong\mathcal{K}_p^{(s)}\oplus\mathcal{K}_{p-1}^{(s)}$$

are valid for $s=1,\cdots,r$ and any integer p.

Proof. The first assertion is evident. For the second one, we take a p-form $\omega\in\mathcal{K}_p^{(s-1)}$, then $i(u_s)_p\omega\in\mathcal{K}_{p-1}(u_s)\cap\mathcal{K}_{p-1}^{(s-1)}=\mathcal{K}_{p-1}^{(s)}$. Conversely, let $\omega\in\mathcal{K}_{p-1}^{(s)}$, and we have by virtue of (2)

$$\omega=(-1)^{s(s-1)/2}i_{1\cdots s}e_{1\cdots s}\omega$$
$$=i(u_s)((-1)^{(s-1)(s+1)/2}i_{1\cdots(s-1)}e_{1\cdots s}\omega)\in\operatorname{Im}(i(u_s)_p|\mathcal{K}_p^{(s-1)}).\quad\text{Q.E.D.}$$

Thus the sequence $\{k_p^{(s)}\}$, ($s=0,1,\cdots,r$; p any) of non-negative integers satisfy

$$k_p^{(s-1)}=k_p^{(s)}+k_{p-1}^{(s)},\tag{3}$$

from which it follows that

$$k_p^{(s)}=\sum_{i=0}^{p}(-1)^i k_{p-i}^{(s-1)}\geqq 0.$$

Especially we have

$$k_p^{(1)}=\sum_{i=0}^{p}(-1)^i b_{p-i}\geqq 0,$$
$$k_p^{(2)}=\sum_{i=0}^{p}(-1)^i k_{p-i}^{(1)}=\sum_{i=0}^{p}(-1)^i(i+1)b_{p-i}\geqq 0.$$

More generally, by making use of the mathematical induction we can prove

Theorem 3.2. *If M^n admits r orthonormal parallel vector fields $u_1,\cdots,u_r$ $(1\leqq r\leqq n)$, then*

$$k_p^{(s)}=\sum_{i=0}^{p}(-1)^i\binom{s+i-1}{i}b_{p-i}\geqq 0$$

are valid for $s=1,\cdots,r$ and any integer p.

Corollary 3.3. *$k_p^{(s)}$ is independent of the choice of s parallel vector fields taken from $u_1, \cdots, u_r$.*

Corollary 3.4. $k_p^{(s)}=0$ *for* $p+s\geqq n+1$.

Especially we have

$$k_{n+1-s}^{(s)}=\sum_{i=0}^{n+1-s}(-1)^i\binom{s+i-1}{i}b_{s+i-1}=0$$

for $s=1, \cdots, r$.

Corresponding to the duality $b_{n-p}=b_p$, the following theorem about $k_p^{(s)}$ holds, by making use of (3) and the mathematical induction.

Theorem 3.5. *If M^n admits r orthonormal parallel vector fields $(1\leqq r\leqq n)$, then we have*

$$k_{n-p}^{(s)}=k_{p-s}^{(s)}$$

for $s=0, 1, \cdots, r$ and any integer p.

References

[1] S. S. Chern: The geometry of G-structure. Bull. Amer. Math. Soc., **72**, 167–219 (1966).
[2] L. Karp: Parallel vector fields and the topology of manifolds. Ibid., **83**, 1051–1053 (1976).
[3] Y. Ogawa and S. Tachibana: On Betti numbers of Riemannian manifolds with parallel vector fields (to appear).