47. Periods of Primitive Forms

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Introduction. We combine Shapiro's lemma on cohomology of groups with Eichler-Shimura isomorphism for elliptic modular forms. As an application of it, we show the rationality of the periods of any primitive cusp form of Neben type. Details will appear elsewhere.

§1. Let Γ be a congruence subgroup of $SL(2,\boldsymbol{Z})$. Γ acts on the complex upper half place H from the left by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z)=(az+b)/(cz+d)$ for $z\in H$. Let $S_{w+2}(\Gamma)$ be the space of cusp forms of weight $w+2\geqq 2$ on Γ, and $S^{\boldsymbol{R}}_{w+2}(\Gamma)$ be the subspace of $S_{w+2}(\Gamma)$ consisting of the cusp forms whose Fourier coefficients at $z=i\infty$ are all real. Let P be the set of all the parabolic elements in $SL(2,\boldsymbol{Z})=\Gamma(1)$. Let $d\vec{z}_w$ be the $(w+1)$ dimensional differential form, the transpose of $(dz, zdz, z^2dz, \cdots, z^w dz)$ on the H. Let ρ_w be the representation of Γ; $\Gamma\to GL(w+1,\boldsymbol{Z})$, which is given by $(cz+d)^{w+2}(d\vec{z}_w\circ g)=\rho_w(g)(d\vec{z}_w)$ for all $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$, where $(d\vec{z}_w)\circ g$ denotes the pull back of $d\vec{z}_w$ by g. Let $\eta_w=\operatorname*{Ind}_{\Gamma\uparrow\Gamma(1)}\rho_w$ be the representation of $\Gamma(1)$ induced from ρ_w. Let $H^1_{P\cap\Gamma}(\Gamma,\rho_w,R)$ and $H^1_P(\Gamma(1),\eta_w,R)$ be the first parabolic cohomology group with R coefficients where $R=\boldsymbol{R}$ or $\boldsymbol{Z}$. $\boldsymbol{R}$, $\boldsymbol{Q}$ and $\boldsymbol{Z}$ denote the real numbers, the rational numbers and the rational integers respectively. Let $g_1=1$, $g_2, g_3, \cdots, g_m$ be representative of the left coset decomposition $\Gamma\backslash\Gamma(1)$. For a $f\in S_{w+2}(\Gamma)$, we set $\mathscr{D}(f)=$the $(w+1)m$ dimensional differential form which is given by $\begin{pmatrix} (f(z)d\vec{z}_w)\circ g_1 \\ (f(z)d\vec{z}_w)\circ g_2 \\ \vdots \\ (f(z)d\vec{z}_w)\circ g_m \end{pmatrix}$, where $(f(z)d\vec{z}_w)\circ g$ denotes the pull back of $(f(z)d\vec{z}_w)$ by $g\in\Gamma(1)$. We normalize η_w such as $\eta_w(g)\mathscr{D}(f)=\mathscr{D}(f)\circ g$. Now let z_0 be any point in the H, $\vec{A}$ be any $(w+1)m$ dimensional column vector in $\boldsymbol{R}^{(w+1)m}$ and w be an arbitrary rational integer$\geqq 0$. Then we have:

Lemma 1. For a $f\in S_{w+2}(\Gamma)$, $\Gamma(1)\ni\sigma\mapsto\operatorname{Re}\int_{z_0}^{\sigma z_0}\mathscr{D}(f)+(\eta_w(\sigma)-1)\vec{A}$ is a cocycle in $Z^1_P(\Gamma(1),\eta_w,\boldsymbol{R})$. Its cohomology class in $H^1_P(\Gamma(1),\eta_w,\boldsymbol{R})$ is determined by f and independent of z_0 and $\vec{A}$.

Theorem 1. *There is an* $\boldsymbol{R}$*-linear surjective isomorphism*

φ; $S_{w+2}(\Gamma) \simeq H^1_P(\Gamma(1), \eta_w, \boldsymbol{R})$ *which is given by* $f \mapsto$ *the cohomology class of* $\left\{\Gamma(1) \ni \sigma \mapsto \mathrm{Re} \int_{z_0}^{\sigma z_0} \mathscr{D}(f)\right\}$.

To prove these we use Shapiro's lemma and Eichler-Shimura isomorphism.

Shapiro's lemma (*e.g.* [5]). *The map* sh; $H^1(\Gamma(1), \eta_w, \boldsymbol{Z}) \rightarrow H^1(\Gamma, \rho_w, \boldsymbol{Z})$ *induced by the compatible maps* $\Gamma \hookrightarrow \Gamma(1)$ *and the projection of* $\boldsymbol{Z}^{(w+1)m}$ *to the first* $(w+1)$ *components is a surjective isomorphism.*

Let sh_P be the restriction of the map sh to $H^1_P(\Gamma(1), \eta_w, \boldsymbol{Z})$. By G. Shimura [13] Proposition 8.6, the natural injection of $Z^1_P(\Gamma(1), \eta_w, \boldsymbol{Z})$ (resp. $Z^1_{P\cap\Gamma}(\Gamma, \rho_w, \boldsymbol{Z})$) into $Z^1_P(\Gamma(1), \eta_w, \boldsymbol{R})$ (resp. $Z^1_{P\cap\Gamma}(\Gamma, \rho_w, \boldsymbol{R})$) induces the $\boldsymbol{R}$-linear surjective isomorphism j_1; $H^1_P(\Gamma(1), \eta_w, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} \boldsymbol{R} \cong H^1_P(\Gamma(1), \eta_w, \boldsymbol{R})$ (resp. j_2; $H^1_{P\cap\Gamma}(\Gamma, \rho_w, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} \boldsymbol{R} \cong H^1_{P\cap\Gamma}(\Gamma, \rho_w, \boldsymbol{R})$). Then we have;

Theorem 2. (i) $sh_P(H^1_P(\Gamma(1), \eta_w, \boldsymbol{Z})) \subset H^1_{P\cap\Gamma}(\Gamma, \rho_w, \boldsymbol{Z})$.

(ii) *The map* $sh_P\boldsymbol{R}: H^1_P(\Gamma(1), \eta_w, \boldsymbol{R}) \rightarrow H^1_{P\cap\Gamma}(\Gamma, \rho_w, \boldsymbol{R})$ *induced by the maps* sh_P, j_1 *and* j_2 *is a surjective* $\boldsymbol{R}$*-linear isomorphism.*

(iii) *The image of* $j_1(H^1_P(\Gamma(1), \eta_w, \boldsymbol{Z}))$ *by the map* $sh_P\boldsymbol{R}$ *coincides with* $j_2(H^1_{P\cap\Gamma}(\Gamma, \rho_w, \boldsymbol{Z}))$.

(iv) *The composite map* $(sh_P\boldsymbol{R}) \circ \varphi$;

$$S_{w+2}(\Gamma) \rightarrow H^1_P(\Gamma(1), \eta_w, \boldsymbol{R}) \rightarrow H^1_{P\cap\Gamma}(\Gamma, \rho_w, \boldsymbol{R})$$

is the Eichler-Shimura isomorphism for $S_{w+2}(\Gamma)$.

We set $E = (sh_P\boldsymbol{R}) \circ \varphi$. As to $E-S$ isomorphism, see [3], [12], [13].

§ 2. Let $N \geqq 1$ be any rational integer. We associate each N with the subgroups $\Gamma_1(N) \subset \Gamma_0(N) \subset SL(2, \boldsymbol{Z})$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \Leftrightarrow c \equiv 0 \bmod N,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \Leftrightarrow a \equiv d \equiv 1 \bmod N \quad \text{and} \quad c \equiv 0 \bmod N.$$

Let χ be any Dirichlet character mod N, w be any rational integer $\geqq 0$, and $S_{w+2}(N, \chi)$ be the space of all the $f(z) \in S_{w+2}(\Gamma_1(N))$ satisfying

$$f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-w-2} = \chi(d) f(z) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

We set $(f \mid [g])(z) = f((az+b)/(cz+d))(cz+d)^{-w-2}$ for $f \in S_{w+2}(\Gamma_1(N))$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{R})$ and $t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Now let F be any primitive form in $S_{w+2}(N, \chi)$ in the sense of Atkin-Lehner [1], Miyake [10], Deligne, Casselman, and W. Li. F has the Fourier expansion $F(z) = \sum_{n=1}^{+\infty} a_n q^n$ where $q = \exp 2\pi i z$ and $a_1 = 1$. We set $\boldsymbol{Q}_F = \boldsymbol{Q}(a_1, a_2, a_3, \cdots) =$ the field generated by all the Fourier coefficients of F over $\boldsymbol{Q}$. Then we have;

Theorem 3. *There are two constants* c^+ *and* c^- *in* $\boldsymbol{C}^{\times}$ *dependent only on* F *such that*

$$\begin{cases} \text{(i)} & \dfrac{1}{c^+}\left\{\displaystyle\int_0^{i\infty} (F\,|\,[g])(z)z^l dz + (-1)^{l+1}\int_0^{i\infty} (F\,|\,[tgt])(z)z^l dz\right\} \in \boldsymbol{Q}_F, \\ \text{(ii)} & \dfrac{1}{c^-}\left\{\displaystyle\int_0^{i\infty} (F\,|\,[g])(z)z^l dz + (-1)^{l}\int_0^{i\infty} (F\,|\,[tgt])(z)z^l dz\right\} \in \boldsymbol{Q}_F \end{cases}$$

for all $g \in SL(2, \boldsymbol{Z})$ *and all rational integers* l *with* $0 \leqq l \leqq w$.

First the result of Theorem 3 type was given in Manin [7] for $S_{w+2}(SL(2, \boldsymbol{Z}))$. Damerell considered the values of a Hecke's L function of imaginary quadratic field based on a different idea [2]. Birch, Manin, Mazur and Swinnerton-Dyer investigated the case of χ=trivial character χ_0, all $a_n \in \boldsymbol{Q}$ and $w+2=2$ in relation to a Weil parametrization ([6], [9]). Shimura investigates the special values of zeta functions associated with a primitive form in connection with the convolution method. Our Corollary 2 of Theorem 3 given below is obtained by him independently of us ([14], [15]). We hear that Razar also proves the Corollary 2 of Theorem 3 in the case $\chi=\chi_0$ under a certain condition on ψ in [11]. Independently of them we proved our Theorem 3 in the case of $\chi=\chi_0$ and any weight in [4] by the period method which is a natural generalization of Manin [7] and is different from the one given in this note and those of Shimura and Razar. Our Theorem 3 and Corollary 1 of Theorem 3 described here are new, not covered by them and not derived from our Corollary 2 of Theorem 3 given below.

For $l \in \boldsymbol{Z}$ with $0 \leqq l \leqq w$ and $x \in \boldsymbol{Q}$, we set $P_l^{\pm}(x) = \dfrac{1}{c^{\pm}}\left\{\displaystyle\int_0^{i\infty} F(z+x)z^l dz \pm (-1)^{l+1}\int_0^{i\infty} F(z-x)z^l dz\right\}$. We have the following two corollaries of Theorem 3.

Corollary 1 of Theorem 3. $P_l^+(x) \in \boldsymbol{Q}_F$ *and* $P_l^-(x) \in \boldsymbol{Q}_F$ *for all* $x \in \boldsymbol{Q}$ *and all rational integers* l *with* $0 \leqq l \leqq w$.

Let ψ be any Dirichlet character, $m(\psi)$ be its conductor and $G(\psi)$ be its Gauss sum $\left(=\sum_{n=1}^{m(\psi)} \psi(n) \exp(2\pi i n/m(\psi))\right)$. We set $F_\psi(z) = \sum_{n=1}^{+\infty} \psi(n) a_n q^n$ where $q = \exp 2\pi i z$ and $\psi(n)=0$ for $(n, m(\psi)) \neq 1$. We set $\boldsymbol{Q}(\psi) = \boldsymbol{Q}(\psi(1), \psi(2), \psi(3), \cdots)$ = the field generated over $\boldsymbol{Q}$ by the values which ψ takes.

Corollary 2 of Theorem 3. *For a rational integer* l *with* $0 \leqq l \leqq w$,

$$\begin{cases} \text{(i)} & \dfrac{1}{c^+ G(\psi)}\displaystyle\int_0^{i\infty} F_\psi(z)z^l dz \in \boldsymbol{Q}_F \cdot \boldsymbol{Q}(\psi) \textit{ for any } \psi \textit{ with } \psi(-1)=(-1)^{l+1}. \\ \text{(ii)} & \dfrac{1}{c^- G(\psi)}\displaystyle\int_0^{i\infty} F_\psi(z)z^l dz \in \boldsymbol{Q}_F \cdot \boldsymbol{Q}(\psi) \textit{ for any } \psi \textit{ with } \psi(-1)=(-1)^{l}. \end{cases}$$

As to the functions $P_l^{\pm}(x)$, we have;

$\sum_{v=0}^{p-1} p^l P_l^{\pm}\left(\dfrac{x+v}{p}\right) = a_p P_l^{\pm}(x) - \chi(p)p^{w-l}P_l^{\pm}(px)$ *for all* $x \in \boldsymbol{Q}$ *and* l *with* $0 \leqq l \leqq w$. *Here we set* $\chi(p)=0$ *for primes* p *with* $p\,|\,N$.

Theorem 3 implies the algebraicity of the p-adic measures $\mu_l^{\pm}$ associated with F on $\left(\varprojlim_m Z/\Delta_0 p^m Z\right)^{\times}$ for an integer Δ_0 with $(p \nmid \Delta_0)$ which are constructed by $P_l^{\pm}(x)$ and the Nasybullin's lemma in Manin [7] 9.4 Lemma. The complex valued measures $\mu_l^{\pm}$ are constructed by B. Mazur, Ju. I. Manin, and Nasybullin in [7], [8], and [9].

To prove the above Theorem 3, we use the following (1)~(5).

(1) The above Theorem 2 for $\Gamma=\Gamma_1(N)$.

(2) The above Theorem 1 for $\Gamma=\Gamma_1(N)$.

(3) $S_{w+2}(\Gamma_1(N))=S_{w+2}^{R}(\Gamma_1(N)) \bigoplus_{R} \sqrt{-1}S_{w+2}^{R}(\Gamma_1(N))$.

(4) For $\Gamma=\Gamma_1(N)$, $\varphi(S_{w+2}^{R}(\Gamma)) \cap j_1(H_P^1(\Gamma(1), \eta_w, Z))$ (resp. $\varphi(\sqrt{-1}S_{w+2}^{R}(\Gamma)) \cap j_1(H_P^1(\Gamma(1), \eta_w, Z))$) is a lattice in $\varphi(S_{w+2}^{R}(\Gamma))$ (resp. $\varphi(\sqrt{-1}S_{w+2}^{R}(\Gamma))$) which is stable by all the Hecke operators on Γ.

(5) Multiplicity one theorem.

Remark. A functional equation $\mu_0^{\pm}(-N^{-1}a^{-1})=-N^{w/2}a^w\tilde{\mu}_0^{\pm}(a)$ is derived at least if $(p, a_p)=1$ and $(N, p\Delta_0)=1$. Here $\tilde{\mu}_0^{\pm}$ denote certain p-adic measures associated with $F|[\omega_N]$.

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