33. A Counterexample to a Conjecture By P. Erdős

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1. Ch. Pommerenke [4] proved the following theorem. Let \( f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n \) be a polynomial of degree \( n \) with some \( a_j \neq 0 \). Assume that the region \( E_f = \{ z \in \mathbb{C} : |f(z)| \leq 1 \} \) is connected, where \( \mathbb{C} \) stands for the field of complex numbers. Then

\[
\max_{z \in E_f} |f'(z)| < \frac{en^2}{2}.
\]

P. Erdős [5] reviewing Pommerenke’s paper conjectured that

\[
\max_{z \in E_f} |f'(z)| < \frac{n^2}{2}
\]

is also true and it is best possible. Erdős reposed his conjecture as a problem in [2]. As it appears in [3] Erdős’ conjecture was unsolved until the year 1972 and to the best of our knowledge it is open until now. The purpose of this paper is to give a counterexample to Erdős’ conjecture. It seems to us that this gives some information concerning the famous coefficient conjecture of L. Bieberbach [1], [6], [7].

2. Counterexample to Erdős’ conjecture. Let \( T_n(z) \) be the Chebyshev polynomial of degree \( n \), defined by \( T_n(z) = 2 \cos n \theta \), where \( z = 2 \cos \theta \), and \( n = 0, 1, 2, 3, \ldots \). This is a complex polynomial of a real variable and has \( n \) real zeros in the line segment \([-2, 2]\) and \(-2 \leq T_n(z) \leq 2 \) for \(-2 \leq z \leq 2\). The recursion formula, \( T_{n+1}(z) = z T_n(z) - T_{n-1}(z) \), which is valid since \( \cos (n+1) \theta + \cos (n-1) \theta = 2 \cos n \theta \cos \theta \), allows us to write the following sequence of polynomials: \( T_0(z) = 2, T_1(z) = z, T_2(z) = z^2 - 2, T_3(z) = z^3 - 3z, T_4(z) = z^4 - 4z^2 + 2 \) and in general

\[
T_n(z) = z^n + \sum_{m=1}^{[n/2]} (-1)^m \frac{n}{m} \left( \frac{n-m-1}{m-1} \right) z^{n-2m}
\]

is a complex inhomogeneous polynomial in a real variable and of degree \( n \). Consider now \( f(z) = \lambda^n T_n(z/\lambda) \). This is a monic inhomogeneous polynomial of degree \( n \) and in fact \(-2 \lambda \leq f(z) \leq 2 \lambda^n \) for \(-2 \lambda \leq z \leq 2 \lambda \). Take \( \lambda = 1/2^{1/n} \). Then \(-1 \leq f(z) \leq 1\) for \(-2/2^{1/n} \leq z \leq 2/2^{1/n}\). Because of the fact that \( T_n(z) = T_n(2 \cos \theta) = 2 \cos n \theta \), it implies that \( T_n'(2 \cos \theta) = (\sin n \theta) / \sin \theta \). Thus, \( \max \{|T_n'(z)| : -2/2^{1/n} \leq z \leq 2/2^{1/n}\} = n^2 \) because \( \max \{|\sin n \theta / \sin \theta| : -2/2^{1/n} \leq z \leq 2/2^{1/n}\} = n \). However, \( f(z) = \lambda^n T_n(z/\lambda) \). Therefore \( f'(z) = \lambda^{n-1} T_n'(z/\lambda) \) and so \( \max \{|f'(z)| : -2 \lambda \leq z \leq 2 \lambda| = \lambda^{n-1} n^2 \). If we set \( \lambda = 1/2^{1/n} \), then \( \max \{|f'(z)| : -2/2^{1/n} \leq z \leq 2/2^{1/n}\} \)}
Claim that $E_f = \{z \in \mathbb{C}: |f(z)| \leq 1\}$ is a connected subset of $\mathbb{C}$. Assume that this is not the case. Then $E_f = A \cup B$ where $A, B$ are disjoint, closed and nonempty subsets of $\mathbb{C}$. It follows that $|f(z)| = 1$ when $z \in \partial A$ (the topological boundary of $A$) by the analyticity of $f$. Thus if $f$ has no zeros in $A$ then the minimum modulus principle implies that $|f(z)| = 1$ in $A$ and which implies that $f(z) = \text{constant}$ on $\mathbb{C}$, which is a contradiction. Hence, $f$ has a zero $x \in A$ and in fact this is a real zero. The same reasoning shows that $f$ has a real zero, $x$, in $B$. Then the closed line segment $[x, x]$ with end points $x, x$ is contained in $E_f = A \cup B$, since $|f(z)| \leq 1$ on the closed real line segment between any two zeros of $f$ which again is a contradiction, for the closed line segment $[x, x]$ is connected and $x \in A, x \in B$ where $A, B$ are disjoint and closed sets in $\mathbb{C}$. Thus $E_f$ is connected. Hence we have given an inhomogeneous polynomial $f(z)$ of degree $n$ with $E_f$ connected subset of $\mathbb{C}$ but $\max_{z \in E_f} |f'(z)| > n^2/2$.

3. Remark. For a better understanding of the set $E_f$ we construct the following figures, as the degree $n$ of the polynomial $f(z)$ varies. Let $n = 2$. Then $T_f(z) = z^2 - 2$, $f(z) = z^2 - 1$. Consider $u(z) = \log |z - 1| + \log |z + 1|$. Then $u(z)$ is a harmonic function on $\mathbb{C} - \{-1, 1\}$. It follows that $u(z) = 0$ on the lemniscate and $u(z) = \infty$ as $|z| = \infty$. Therefore $u(z) > 0$ outside the lemniscate. It is clear that $u(z) < 0$ inside the lemniscate. The picture of $E_f$ is the shadowed region in Fig. 1, and $\{z \in \mathbb{C}: |f(z)| = 1\} = \{-2, 0, 2\}$.

![Fig. 1](image)

Similarly, working for $n = 3$ we find for $E_f$ the shadowed region given by Fig. 2, and for $n = 4$, we find for $E_f$ the shadowed region given by Fig. 3. In a similar manner we obtain the figures for $E_f$, as $n \geq 5$.

4. Open problem. Find the least upper bound of the $\max_{z \in E_f} |f'(z)|$?

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References


