How often can two independent elephant random walks on Z meet?

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Abstract: We show that two independent elephant random walks on the integer lattice **Z** meet each other finitely often or infinitely often depends on whether the memory parameter p is strictly larger than $3/4$ or not. Asymptotic results for the distance between them are also obtained.

Key words: Self-interacting random walks; Random walks with memory; Elephant random walks.

1. Introduction and results. Over 100 years have passed since the term "random walk" appeared in the letter of Pearson [9]. It is arguably the simplest stochastic process. Pólya [10, 11] started the study of random walk on the d-dimensional lattice: A wandering point (or a random walker) moves uniformly at random between the nearest-neighbour sites on the integer lattice $\mathbf{Z}^d$ with each step being taken independent of past steps. As is explained in Pólya [12], his main motivation was to understand whether the probability that two independent random walkers eventually meet each other is one or not. Problems on random walkers *with memory*, namely their future evolution depend on the entire history of the process, appeared more recently in connection with several applications. Such examples include self-avoiding random walks, reinforced random walks, and elephant random walks. See Hughes [6], Révész [14], Laulin [8], and the references therein.

A similar problem described in the previous paragraph arises also for two independent random walkers with memory. In this note we give an answer for the elephant random walk, introduced by Schütz and Trimper [15]. The first step X_1 of the walker is $+1$ with probability s, and -1 with probability $1-s$. For each $n = 1, 2, \cdots$, let U_n be uniformly distributed on $\{1, \cdots, n\}$, and

$$X_{n+1} = \begin{cases} X_{U_n} & \text{with probability } p, \\ -X_{U_n} & \text{with probability } 1-p, \end{cases}$$

where $\{U_n : n = 1, 2, \cdots\}$ is an independent family of random variables. The sequence $\{X_i\}$ generates a one-dimensional random walk $\{S_n\}$ by

$$S_0 := 0, \quad \text{and} \quad S_n = \sum_{i=1}^{n} X_i \quad \text{for } n = 1, 2, \cdots.$$

In the case $p = 1/2$, $\{S_n\}$ is essentially the usual symmetric random walk. For $p > 1/2$ [resp. $p < 1/2$] the walker prefers to do the same as [resp. the opposite of] the previous decision.

Schütz and Trimper [15] show that there are two distinct (diffusive/super-diffusive) regimes about the asymptotic behavior of the mean square displacement. After their study, several limit theorems describing the influence of the memory parameter p have been studied by many authors [1–5, 7, 13]:

(a) When $0 < p < 3/4$ the elephant random walk is diffusive, and the fluctuation is Gaussian:

$$\frac{S_n}{\sqrt{n}} \xrightarrow{\text{d}} N\left(0, \frac{1}{3-4p}\right) \quad \text{as } n \to \infty.$$

where $\xrightarrow{\text{d}}$ denotes the convergence in distribution, and $N(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2.

(b) When $p = 3/4$ the walk is marginally superdiffusive and

$$\frac{S_n}{\sqrt{n \log n}} \xrightarrow{\text{d}} N(0, 1) \quad \text{as } n \to \infty.$$

(c) If $3/4 < p < 1$ then there exists a random variable L with a continuous distribution depending on both p and s such that

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$$(\star) \qquad \frac{S_n}{n^{2p-1}} \to L \quad \text{a.s. and in } L^2 \text{ as } n \to \infty.$$

Although L is non-Gaussian, the fluctuation from the "random drift" Ln^{2p-1} due to the strong memory effect is still Gaussian:

$$\frac{S_n - Ln^{2p-1}}{\sqrt{n}} \xrightarrow{\mathrm{d}} N\left(0, \frac{1}{4p-3}\right) \quad \text{as } n \to \infty.$$

Our main result is the following

Theorem 1.1. *If $0 < p \leq 3/4$ then two independent elephant random walks with the same memory parameter p meet each other infinitely often with probability one. On the other hand, if $3/4 < p < 1$ then they meet each other only finitely often with probability one.*

This theorem shows that two elephant random walks cannot meet infinitely often if the memory effect is too strong.

Theorem 1.1 follows from

Theorem 1.2. *Let $\{S_n\}$ and $\{S'_n\}$ be two independent elephant random walks with the same memory parameter p.*

(i) *If $0 < p < 3/4$ then*

$$\limsup_{n\to\infty} \pm \frac{S_n - S'_n}{\sqrt{n \log\log n}} = \frac{2}{\sqrt{3-4p}} \quad a.s.$$

(ii) *If $p = 3/4$ then*

$$\limsup_{n\to\infty} \pm \frac{S_n - S'_n}{\sqrt{n \log n \log\log\log n}} = 2 \quad a.s.$$

(iii) *If $3/4 < p < 1$ then*

$$\lim_{n\to\infty} \frac{S_n - S'_n}{n^{2p-1}} = M \quad a.s.,$$

where M is a random variable with

$$P(M \neq 0) = 1.$$

2. Proof of Theorem 1.2. We use the following strong approximation result.

Lemma 2.1 (Coletti, Gava and Schütz [4]). *Let $\{S_n\}$ be the elephant random walk with the memory parameter p, and $\{B(t)\}$ be the standard Brownian motion.*

(i) *If $0 < p < 3/4$ then*

$$S_n - \frac{n^{2p-1}}{\sqrt{3-4p}} \cdot B(n^{3-4p}) = o(\sqrt{n \log\log n}) \quad a.s.$$

(ii) *If $p = 3/4$ then*

$$S_n - \sqrt{n} \cdot B(\log n) = o(\sqrt{n \log n \log\log\log n}) \quad a.s.$$

Let $\{B(t)\}$ and $\{B'(t)\}$ be two independent standard Brownian motions. It is straightforward to see that $\{B(t) - B'(t)\}$ and $\{\sqrt{2}B(t)\}$ have the same distribution. By the law of the iterated logarithm for the standard Brownian motion,

$$\limsup_{t\to\infty} \frac{B(t) - B'(t)}{\sqrt{t \log\log t}} = 2$$

with probability one. Thus we have

$$\limsup_{n\to\infty} \frac{n^{2p-1}\{B(n^{3-4p}) - B'(n^{3-4p})\}}{\sqrt{n \log\log n}} = 2$$

and

$$\limsup_{n\to\infty} \frac{\sqrt{n}\{B(\log n) - B'(\log n)\}}{\sqrt{n \log n \log\log\log n}} = 2$$

with probability one. Now Theorem 1.2 (i) [resp. (ii)] follows from Lemma 2.1 (i) [resp. (ii)].

Now we turn to the case $3/4 < p < 1$. By $(\star)$,

$$\lim_{n\to\infty} \frac{S_n}{n^{2p-1}} = L \quad \text{and} \quad \lim_{n\to\infty} \frac{S'_n}{n^{2p-1}} = L' \quad \text{a.s.}$$

Noting that L and L' are independent, and both of them have continuous distributions, we have that $P(L = L') = 0$. Putting $M := L - L'$, we obtain Theorem 1.2 (iii). □

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References

[1] E. Baur and J. Bertoin, Elephant random walks and their connection to Pólya-type urns, Phys. Rev. E **94** (2016), 052134.

[2] B. Bercu, A martingale approach for the elephant random walk, J. Phys. A **51** (2018), no. 1, 015201.

[3] C. F. Coletti, R. J. Gava and G. M. Schütz, Central limit theorem for the elephant random walk, J. Math. Phys. **58** (2017), 053303.

[4] C. F. Coletti, R. J. Gava and G. M. Schütz, A strong invariance principle for the elephant random walk, J. Stat. Mech. Theory Exp. (2017), no. 12, 123207.

[5] H. Guérin, L. Laulin and K. Raschel, A fixed-point equation approach for the superdiffusive elephant random walk, arXiv:2308.14630.

[6] B. D. Hughes, *Random walks and random environments. Vol. 1*, Oxford Science Publications, Clarendon Press, Oxford, 1995.

[7] N. Kubota and M. Takei, Gaussian fluctuation for superdiffusive elephant random walks, J. Stat. Phys. **177** (2019), no. 6, 1157–1171.

[8] L. Laulin, Autour de la marche aléatoire de l'éléphant (About the elephant random walk), thèse, Université de Bordeaux, HAL Id: tel-03708512.

[9] K. Pearson, The problem of the random walk, Nature **72** (1905), 294.

[10] G. Pólya, Quelques problèmes de probabilité se rapportant à la "promenade au hazard", Ens. math. **20** (1919), 444–445.

[11] G. Pólya, Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz, Math. Ann. **84** (1921), no. 1–2, 149–160.

[12] G. Pólya, Two incidents, in *Scientists at Work: Festschrift in Honour of Herman Wold,* Almquist & Wiksells Boktryckeri AB, Stockholm, 1970, pp. 165–168. Reprinted in his Collected Papers, Volume IV (1984), 582–585.

[13] S. Qin, Recurrence and transience of multidimensional elephant random walks, arXiv: 2309.09795.

[14] P. Révész, *Random walk in random and non-random environments*, 3rd ed., World Scientific, New Jersey, 2013.

[15] G. M. Schütz, and S. Trimper, Elephants can always remember: Exact long-range memory effects in a non-Markovian random walk, Phys. Rev. E **70** (2004), 045101.