Hook formula for Coxeter groups via the twisted group ring

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Abstract: We use Kostant and Kumar's twisted group ring and its dual to formulate and prove a generalization of Nakada's colored hook formula for any Coxeter groups. For dominant minuscule elements of the Weyl group of a Kac–Moody algebra, this provides another short proof of Nakada's colored hook formula.

Key words: Hook formula; Coxeter group; Kostant and Kumar's twisted group ring.

1. Introduction. The purpose of this note is to give a short algebraic proof of Nakada's colored hook formula (Corollary 5.1) and its generalization to any Coxeter groups (Theorem 5.1). As a by-product of our formulation, we get a simple proof of Shi's Yang–Baxter relations [11] in the group algebra $Q(V)[W]$ for a Coxeter group W (see Remark 4.1 below), where V is the underlying vector space of the root datum of W, see Section 2.1.

For the proof, we use Kostant–Kumar's twisted group ring H_Q (see 3.1), $\mathcal{L}_w \in H_Q$ (cf. Definition1), the dual basis η^w (cf. Equation 9), and a Molev–Sagan recursion formula (17) (cf. [9, Prop. 3.2]). The $\mathcal{L}_w$ and η^w are algebraic counterparts of some geometric objects studied in [8]. In Appendix 6, we give a brief explanation of the geometric background of this construction in the finite Weyl group case.

2. Coxeter group and root system. In this section, we recall some fundamental properties of the Coxeter group and its root system.

2.1. Coxeter group and root system. Let (W,S) be a Coxeter system, where $S=\{s_i\}_{i\in I}$ is the set of generators. For any $w\in W$, the support of w (the set of generators in S which appear in some reduced expression of w) is a finite set. Hence, for the purpose of generalizing the hook formula (Theorem 5.1), we can assume $I:=\{1,2,\dots,r\}$ is a finite set. Let $(V,\Sigma,\Sigma^\vee)$ be a triple (called the root datum of W) with the following properties:

(R_0) V is a finite-dimensional vector space over $\mathbf{R}$ which is a representation space of W. Let $V^*=\mathrm{Hom}_{\mathbf{R}}(V,\mathbf{R})$, with the natural pairing $(\cdot,\cdot): V\times V^*\to\mathbf{R}$.

(R_1) $\Sigma=\{\alpha_1,\dots,\alpha_r\}\subset V$ and $\Sigma^\vee=\{\alpha_1^\vee,\dots,\alpha_r^\vee\}\subset V^*$. The elements in Σ and $\Sigma^\vee$ are assumed to be linearly independent.

(R_2) $(\alpha_i,\alpha_i^\vee)=2$ for $i=1,2,\dots,r$.

(R_3) W acts on V by $s_i\lambda=\lambda-(\lambda,\alpha_i^\vee)\alpha_i$ for $\lambda\in V$ and $i=1,2,\dots,r$.

Note that such a triple $(V,\Sigma,\Sigma^\vee)$ always exists. For example, for a Coxeter system (W,S), a "Cartan" matrix $C=(c_{i,j})_{r\times r}$ whose entry is presumed to be $c_{i,j}=(\alpha_j,\alpha_i^\vee)$, can defined as follows: Let $M=(m_{i,j})_{r\times r}$ be the Coxeter matrix of (W,S) whose entry $m_{i,j}$ is the order of s_is_j. By definition, $m_{i,j}=m_{j,i}\in\{1,2,3,\dots\}\cup\{\infty\}$ and $m_{i,i}=1$ for $1\le i,j\le r$. We define a "Cartan" matrix $C=(c_{i,j})_{r\times r}$ whose entry $c_{i,j}$ is a real number satisfying the following conditions.

(1) $c_{i,i}=2$ $(1\le i\le r)$.
(2) if $i\ne j$ then $c_{i,j}\le 0$, and $c_{i,j}=0$ if $m_{i,j}=2$.
(3) for $m_{i,j}>2$, it is required that

$$c_{i,j}c_{j,i}=4\cos\left(\frac{\pi}{m_{i,j}}\right)^2.$$

For a crystallographic Coxeter group W ($m_{i,j}\in\{1,2,3,4,6,\infty\}$ for all $1\le i,j\le r$), such as the Weyl group of a Kac–Moody algebra, we can take all $c_{i,j}$ to be integers. For a standard choice of an arbitrary Coxeter group, we can take C to be symmetric ($c_{i,j}=c_{j,i}=-2\cos(\pi/m_{i,j})$). Then using the argument of [5, Proposition 1.1] over the real numbers, we have such a triple $(V,\Sigma,\Sigma^\vee)$ for (W,S).

Let $R=\{w(\alpha_i)\mid w\in W, i=1,\dots,r\}\subset V$ be the set roots of (W,S). It is known that we have a disjoint union $R=R^+\sqcup R^-$, where $R^+=\{\alpha\in R\mid$

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$\alpha = \sum_{i=1}^{r} c_i\alpha_i, c_i \geq 0\}$ is the set of positive roots and $R^- = -R^+$ (cf. [2] for basic properties of the root system of a Coxeter group.). The action of W on V^* is defined by

$$s_i(y) = y - (\alpha_i, y)\alpha_i^\vee \text{ for } y \in V^* \text{ for } i = 1, 2, \ldots, r.$$

For a root $\gamma \in R$, the dual root $\gamma^\vee \in V^*$ is defined by $\gamma^\vee = w(\alpha_i^\vee)$ if $\gamma = w(\alpha_i)$. For a positive root $\beta \in R^+$, let $s_\beta \in W$ be the corresponding reflection, i.e.

$$s_\beta(x) = x - (x, \beta^\vee)\beta \text{ for } x \in V,$$
$$s_\beta(y) = y - (\beta, y)\beta^\vee \text{ for } y \in V^*.$$

The pairing $(\cdot,\cdot)$ is W-invariant, i.e.

$$(w(x), w(y)) = (x, y) \text{ for all } x \in V, y \in V^*, w \in W.$$

2.2. Some lemmas. In this section, we list some results about the Coxeter group and the root system. For any $w \in W$, let $S(w) := R^+ \cap wR^-$.

Lemma 2.1. *For a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$, let*

$$\beta_j := s_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j} \quad (1 \leq j \leq \ell). \tag{1}$$

Then $|S(w)| = \ell$ and $S(w) = \{\beta_1, \beta_2, \ldots, \beta_\ell\}$.

Proof. This is a consequence of [4, Theorem 5.4], i.e., for $u \in W$, $\ell(us_i) > \ell(u) \iff u(\alpha_i) > 0$. □

Remark 2.1. By this Lemma, we see that the representation of W on V is faithful.

Let $J \subset S$ be a subset of generators of W and W_J be the subgroup generated by $\{s_i\}_{i\in J}$. Let $W^J := W/W_J$ be the set of minimal length coset representatives. Let us denote by $<$ the Bruhat order on W ([4, 5.9]). Then it induces a Bruhat order on W^J. An element $\chi \in V$ is said to be dominant if $(\chi, \alpha_j^\vee) \geq 0$ for any $1 \leq j \leq r$.

Lemma 2.2 ([1, Ch.5, §4.6])**.** *If $\chi \in V$ is dominant, the stabilizer subgroup $stab_W(\chi) := \{w \in W \mid w(\chi) = \chi\}$ is equal to W_J for some $J \subset S$, i.e., for $x \neq y \in W^J$, $x(\chi) - y(\chi) \neq 0$.*

Lemma 2.3 ([3, Lemma 4.1])**.** *Asume $\chi \in V$ is dominant with $stab_W(\chi) = W_J$. For $x < y \in W^J$, the two conditions below are equivalent.*

(a) $\exists\gamma \in R^+$ such that $yW_J = xs_\gamma W_J$,

(b) $\exists\beta \in R^+$ such that $yW_J = s_\beta xW_J$.

Moreover, if these conditions are satisfied, then

$$x(\chi) - y(\chi) = (\chi, \gamma^\vee)\beta.$$

In particular, β is unique if it exists.

3. Twisted group ring and its dual.

3.1. Kostant–Kumar twisted group ring $\boldsymbol{H_Q}$. Assume (W, S) is a Coxeter system with root datum $(V, \Sigma, \Sigma^\vee)$ (cf. Section 2). The Coxeter group analog of the Kostant–Kumar's twisted group algebra is $H_Q := Q(V) \rtimes \mathbf{R}[W]$, the smash product of the fraction field $Q(V)$ of the symmetric algebra $S(V)$ with the group algebra $\mathbf{R}[W]$. As a vector space over $\mathbf{R}$, $H_Q = Q(V) \otimes_{\mathbf{R}} \mathbf{R}[W]$, with $Q(V)$-free basis $\{\delta_w\}_{w\in W}$, and the multiplication is defined as follows. For $a = \sum_{w\in W} a_w\delta_w, b = \sum_{u\in W} b_u\delta_u \in H_Q$,

$$a \cdot b = \sum_{w,u\in W} a_w w(b_u)\delta_{wu}.$$

If W is the Weyl group of a Kac–Moody algebra with an integral Cartan matrix, $H_Q \otimes \mathbf{C}$ is the twisted group ring introduced by Kostant–Kumar [6], where the divided difference operator $\partial_i = \frac{1}{\alpha_i}\delta_{s_i} - \frac{1}{\alpha_i}\delta_{id}$ and the associated basis element $\partial_w \in H_Q$ $(w \in W)$ are defined. Here we introduce another (inhomogeneous) basis $\mathcal{L}_w$, which is the main tool of our calculation.

Definition 1. For $i = 1, 2, \ldots r$, define $\mathcal{L}_i \in H_Q$ by $\mathcal{L}_i := \frac{1+\alpha_i}{\alpha_i}\delta_{s_i} - \frac{1}{\alpha_i}\delta_{id}$, i.e. $\mathcal{L}_i = \partial_i + \delta_{s_i}$.

From the definition, $\delta_{id} + \alpha_i\mathcal{L}_i = (1+\alpha_i)\delta_{s_i}$ and $\mathcal{L}_i^2 = \delta_{id}$.

Proposition 3.1. *For $w \in W$, the following holds.*

(a)$_w$ *Let $w = s_{i_1}\cdots s_{i_\ell} \in W$ be a reduced expression, and define $\mathcal{L}_w := \mathcal{L}_{i_1}\cdots\mathcal{L}_{i_\ell}$. Then it does not depend on the choice of the reduced expression of w.*

(b)$_w$ *For $\chi \in V \subset S(V)$, the following equality holds. (We abbreviate $\chi\delta_{id}$ as χ.)*

$$\mathcal{L}_w\chi = w(\chi)\mathcal{L}_w - \sum_{\gamma\in S(w^{-1})} (\chi, \gamma^\vee)\mathcal{L}_{ws_\gamma}. \tag{2}$$

(c)$_w$ *Let $\mathcal{L}_w = \sum_{v\in W} e_{w,v}\delta_v$, $e_{w,v} \in Q(V)$. Then $e_{w,v} = 0$ unless $v \leq w$, and*

$$e_{w,w} = \prod_{\beta\in S(w)} \frac{1+\beta}{\beta}.$$

Proof. We will prove (a)$_w$,(b)$_w$,(c)$_w$ simultaneously, by induction on length $\ell(w)$ of w as in [6]. If $\ell(w) = 1$, i.e., $w = s$ for $s \in S$, we can check the formulae (a)$_s$,(b)$_s$,(c)$_s$ directly. Assume $\ell(w) > 1$ and $u = sw < w$, $s \in S$. Then we have, by (b)$_u$ and (b)$_s$, $\mathcal{L}_s\mathcal{L}_u\chi =$

$$w(\chi)\mathcal{L}_s\mathcal{L}_u - (u(\chi), \alpha_s^\vee)\mathcal{L}_u - \sum_{\gamma\in S(u^{-1})} (\chi, \gamma^\vee)\mathcal{L}_s\mathcal{L}_{us_\gamma}.$$

As $S(w^{-1}) = S(u^{-1}) \cup \{u^{-1}\alpha_s\}$, and $(a)_z$ for elements z of length less than $\ell(w)$, we have

$$\mathcal{L}_s\mathcal{L}_u\chi - w(\chi)\mathcal{L}_s\mathcal{L}_u = -\sum_{\gamma\in S(w^{-1})}(\chi,\gamma^\vee)\mathcal{L}_{ws_\gamma},$$

which gives part $(b)_w$. Likewise, for $s' \in S$ with $v = s'w < w$, we have

$$\mathcal{L}_{s'}\mathcal{L}_v\chi - w(\chi)\mathcal{L}_{s'}\mathcal{L}_v = -\sum_{\gamma\in S(w^{-1})}(\chi,\gamma^\vee)\mathcal{L}_{ws_\gamma}.$$

Therefore,

$$\mathcal{L}_s\mathcal{L}_u\chi - w(\chi)\mathcal{L}_s\mathcal{L}_u = \mathcal{L}_{s'}\mathcal{L}_v\chi - w(\chi)\mathcal{L}_{s'}\mathcal{L}_v. \tag{3}$$

Write $\mathcal{L}_s\mathcal{L}_u = \sum_{x\in W} q_x\delta_x$, and $\mathcal{L}_{s'}\mathcal{L}_v = \sum_{x\in W} q'_x\delta_x$ ($q_x, q'_x \in Q(V)$). Then by $(c)_u$ and $(c)_v$, we have $q_x = q'_x = 0$ unless $x \le w$ and

$$q_w = \prod_{\beta\in S(w)}\frac{1+\beta}{\beta} = q'_w, \tag{4}$$

which gives part $(c)_w$. From (3) we have

$$(x(\chi) - w(\chi))q_x = (x(\chi) - w(\chi))q'_x \text{ for } \forall x \in W.$$

As V is faithful (Remark 2.1), we have $q_x = q'_x$ for $x \ne w$. Together with (4), we have

$$\mathcal{L}_s\mathcal{L}_u = \mathcal{L}_{s'}\mathcal{L}_v,$$

which proves part $(a)_w$. □

By this Proposition, $\{\mathcal{L}_w\}_{w\in W}$ forms a basis of the left $Q(V)$-module H_Q, and if we expand $\mathcal{L}_w\chi$ in this basis

$$\mathcal{L}_w\chi = \sum_{v\in W} c^w_{\chi,v}\mathcal{L}_v, \quad c^w_{\chi,v} \in Q(V), \chi \in V, \tag{5}$$

we have

$$c^w_{\chi,v} = \begin{cases} w(\chi) & \text{if } v = w \\ -(\chi,\gamma^\vee) & \text{if } v < w = vs_\gamma, \gamma \in R^+ \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

3.2. Dual basis $\{\eta^w\}_{w\in W}$ of $\{\mathcal{L}_w\}_{w\in W}$. Let $\mathrm{Fun}(W, Q(V))$ denote the ring of functions on W with values in $Q(V)$, with natural $Q(V)$-module structure by $(q\xi)(w) = q\xi(w)$ for $q \in Q(V), w \in W$, and $\xi \in \mathrm{Fun}(W, Q(V))$. There is a perfect pairing $\langle\cdot,\cdot\rangle : H_Q \times \mathrm{Fun}(W, Q(V)) \to Q(V)$, given by

$$\langle a, \xi\rangle = \sum_{w\in W} a_w\xi(w), \tag{7}$$

for $a = \sum_{w\in W} a_w\delta_w \in H_Q$ and $\xi \in \mathrm{Fun}(W, Q(V))$. Here a pairing $\langle,\rangle : M_1 \times M_2 \to Q(V)$ is perfect if it induces an isomorphism $M_1^* \simeq M_2$ of $Q(V)$-modules.

Lemma 3.1. *For $\chi \in V$, define $L_\chi \in \mathrm{Fun}(W, Q(V))$ by $L_\chi(w) := w(\chi)$. Then we have*

$$\langle h\chi, f\rangle = \langle h, L_\chi f\rangle \tag{8}$$

for $h \in H_Q, f \in \mathrm{Fun}(W, Q(V))$.

Proof. If $h = \sum_{w\in W} a_w\delta_w$, then $h\chi = \sum_{w\in W} a_w w(\chi)\delta_w$. Therefore we have $\langle h\chi, f\rangle = \sum_{w\in W} a_w w(\chi)f(w) = \sum_{w\in W} a_w(L_\chi f)(w) = \langle h, L_\chi f\rangle$. □

We can define element $\eta^v \in \mathrm{Fun}(W, Q(V))$ for $v \in W$ by duality

$$\langle\mathcal{L}_w, \eta^v\rangle = \delta_{w,v} \qquad \text{for } \forall w \in W. \tag{9}$$

Using this duality and Proposition 3.1 $(c)_w$, we have

(10)
$$\eta^v(w) = 0 \text{ unless } v \le w, \text{ and } \eta^w(w) = \prod_{\beta\in S(w)}\frac{\beta}{1+\beta}.$$

Let $J \subset S$ be a subset of the generators of W.

Definition 2. For $v \in W^J$, define $\eta^v_J \in \mathrm{Fun}(W^J, Q(V))$ by

$$\eta^v_J(w) := \sum_{u\in vW_J, u\le w}\eta^u(w) \text{ for } w \in W^J. \tag{11}$$

We can formally write $\eta^v_J = \sum_{u\in vW_J}\eta^u$, as $\eta^u(w) = 0$ if $u \not\le w$. From (10) it follows that for $v, w \in W^J$,

(12)
$$\eta^v_J(w) = 0 \text{ unless } v \le w, \text{ and } \eta^w_J(w) = \prod_{\beta\in S(w)}\frac{\beta}{1+\beta}.$$

Thus, $\{\eta^w_J \mid w \in W^J\}$ is a basis for $\mathrm{Fun}(W^J, Q(V))$ over $Q(V)$.

For any $v, z \in W^J$ and $\chi \in V^{W_J}$, define

$$c^{z,J}_{\chi,v} := \sum_{u\in W_J} c^{zu}_{\chi,v}. \tag{13}$$

By the equalities (6) and (13), we have the following equality.

Lemma 3.2.

$$c^{z,J}_{\chi,v} = \begin{cases} v\chi, & \text{if } v = z; \\ -(\chi,\gamma^\vee), & \text{if } vW_J < vs_\gamma W_J = zW_J, \gamma \in R^+; \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3.2. *For $\chi \in V^{W_J}$, define $L_\chi^J \in \mathrm{Fun}(W^J, Q(V))$ by $L_\chi^J(z) = z(\chi)$ for $z \in W^J$. Then for any $v \in W^J$,*

$$(L_\chi^J \eta_J^v)(w) = \sum_{z\in W^J, v\le z\le w} c_{\chi,v}^{z,J}\eta_J^z(w), \text{ for } w \in W^J. \tag{14}$$

We can formally write $L_\chi^J \eta_J^v = \sum_{z\in W^J, v\le z} c_{\chi,v}^{z,J}\eta_J^z$.

Proof. Combining duality (9), Equation (5) and Lemma 3.1, we get

$$(L_\chi \eta^v)(w) = \sum_{z\in W, v\le z\le w} c_{\chi,v}^{z}\eta^z(w), \text{ for } w \in W. \tag{15}$$

Moreover, it is easy to see that $c_{\chi,vu}^{zy} = c_{\chi,v}^{zyu^{-1}}$ for any $z, v \in W^J$ and $y, u \in W_J$. Then Equation (14) readily follows by these observations. □

For $u, v, w \in W^J$, let $d_{u,v}^{w,J} \in Q(V)$ be the structure constants, i.e.,

$$\eta_J^u \eta_J^v = \sum_{w\in W^J} d_{u,v}^{w,J}\eta_J^w. \tag{16}$$

Lemma 3.3. *The coefficients $d_{u,v}^{w,J} \in Q(V)$ have the following properties.*

(i) *For $u, v, w \in W^J$, $d_{u,v}^{w,J} = 0$ unless $u \le w$ and $v \le w$.*

(ii) *For $v, w \in W^J$, $d_{v,w}^{w,J} = \eta_J^v(w)$,*

(iii) *For $w \in W^J$, $d_{w,w}^{w,J} = \prod_{\beta\in S(w)} \dfrac{\beta}{1+\beta}$.*

Proof. Given $u, v \in W^J$, if there is a $w \in W^J$ such that $u \not\le w$ or $v \not\le w$ and $d_{u,v}^{w,J} \neq 0$, take a minimal such w in Bruhat order and evaluate both sides of (16) at w. The left hand side becomes 0, but right hand side is nonzero by (12), which gives a contradiction. Therefore (i) holds. As $\eta_J^v(w)\eta_J^w(w) = d_{v,w}^{w,J}\eta_J^w(w)$, and $\eta_J^w(w) = \eta^w(w) \neq 0$, we have (ii). Finally, (iii) follows from Equation (12). □

Proposition 3.3. *For any $u, v, w \in W^J$ and any vector $\chi \in V^{W_J}$, the following holds:*

$$(c_{\chi,w}^{w,J} - c_{\chi,u}^{u,J})d_{u,v}^{w,J} = \sum_{u<x\le w, x\in W^J} c_{\chi,u}^{x,J} d_{x,v}^{w,J} - \sum_{u,v\le y<w, y\in W^J} c_{\chi,y}^{w,J} d_{u,v}^{y,J}.$$

Proof. By taking the coefficient of η_J^w in $L_\chi(\eta_J^u\eta_J^v) = (L_\chi\eta_J^u)\eta_J^v$, we have

$$c_{\chi,w}^{w,J} d_{u,v}^{w,J} + \sum_{u,v\le y<w, y\in W^J} c_{\chi,y}^{w,J} d_{u,v}^{y,J}$$
$$= c_{\chi,u}^{u,J} d_{u,v}^{w,J} + \sum_{u<x\le w, x\in W^J} c_{\chi,u}^{x,J} d_{x,v}^{w,J},$$
from which the assertion holds. □

Corollary 3.1. *If $\chi \in V^{W_J}$ satisfies $c_{\chi,w}^{w,J} \neq c_{\chi,u}^{u,J}$ (e.g. $\chi = \pi_J$, cf. Lemma 2.2.), then we have*

$$d_{u,v}^{w,J} = \frac{1}{c_{\chi,w}^{w,J} - c_{\chi,u}^{u,J}} \times \left(\sum_{u<x\le w, x\in W^J} c_{\chi,u}^{x,J} d_{x,v}^{w,J} - \sum_{u,v\le y<w, y\in W^J} c_{\chi,y}^{w,J} d_{u,v}^{y,J}\right).$$

In particular, for the case $v = w$,

$$d_{u,w}^{w,J} = \sum_{x\in W^J, u<x\le w} \frac{c_{\chi,u}^{x,J}}{c_{\chi,w}^{w,J} - c_{\chi,u}^{u,J}} d_{x,w}^{w,J}. \tag{17}$$

4. Yang–Baxter elements in the group algebra $\boldsymbol{Q(V)[W]}$. Let $Q(V)[W]$ denote the group algebra of W over $Q[V]$. The purpose of this section is to relate H_Q with $Q(V)[W]$ through a left $Q(V)$-module homomorphism and prove the equations in Corollary 4.1. As a byproduct, we get a simple proof for the Yang–Baxter relations for the Coxeter groups ([11, Proposition4.1]).

Let $\Delta_i := (1+\alpha_i)\delta_{s_i} = 1 + \alpha_i\mathcal{L}_i \in H_Q$. For a reduced expression $w = s_{i_1}\cdots s_{i_\ell}$, define

$$\Delta_w := \Delta_{i_1}\Delta_{i_2}\cdots\Delta_{i_\ell} \in H_Q. \tag{18}$$

Then it is easy to see that $\Delta_w = A(w)\delta_w$, where $A(w) = \prod_{j=1}^{\ell}(1+\beta_j)$ for β_j defined in (1). By Lemma 2.1, Δ_w does not depend on the choice of the reduced expression for w. We can expand Δ_w in terms of $\{\mathcal{L}_v\}_{v\in W}$ as follows:

$$\Delta_w = \sum_{v\in W} q(v,w)\mathcal{L}_v, \quad q(v,w) \in Q(V). \tag{19}$$

Lemma 4.1. *The coefficients $q(v, w)$ satisfy the following properties.*

(i) *$q(v,w) = A(w)\eta^v(w)$.*

(ii) *$q(v,w) = 0$ unless $v \le w$.*

(iii) *If $s_iw > w$,*

$$q(u, s_iw) = s_i(q(u,w)) + \alpha_i s_i(q(s_iu, w)). \tag{20}$$

Proof. (i) follows by evaluation of $\langle\Delta_w, \eta^v\rangle$, using definitions (7) and (9). (ii) follows from (i) and (10). If $s_iw > w$, by comparing the coefficient of $\mathcal{L}_u$ in

$$\Delta_{s_iw} = \Delta_{s_i}\Delta_w = (1+\alpha_i)\delta_{s_i}\sum_{u\in W} q(u,w)\mathcal{L}_u$$
$$= \sum_{u\in W} s_i(q(u,w))(1+\alpha_i\mathcal{L}_i)\mathcal{L}_u,$$

we get the relation (iii), as $(\mathcal{L}_i)^2 = \delta_{id}$. □

We can now define a left $Q(V)$-module isomorphism $\Phi : H_Q \to Q(V)[W]$ by sending

$$\Phi\left(\sum_{w\in W} c_w \mathcal{L}_w\right) = \sum_{w\in W} c_w w, \ \ c_w \in Q(V).$$

Definition 3. For $w \in W$, define a Yang–Baxter element $Y_w := \Phi(\Delta_w) \in Q(V)[W]$.

Then by (19),

$$Y_w = \sum_{v\le w} q(v,w)v. \tag{21}$$

Proposition 4.1. [11] *For a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_\ell} \in W$, let β_j be as in* (1)*. Then the following equality holds in $Q(V)[W]$.*

$$Y_w = (1+\beta_1 s_{i_1})(1+\beta_2 s_{i_2})\cdots(1+\beta_\ell s_{i_\ell}). \tag{22}$$

Proof. It follows directly by induction on $\ell(w)$ and Lemma 4.1 (iii). □

Remark 4.1. The proof of the above Proposition gives a simple proof for [11, Theorem 3.2], by replacing α_i with $-\alpha_i$ for all $i \in I$.

Corollary 4.1. *For $w \in W$, we have*

$$A(w) = \sum_{v\le w} q(v,w), \tag{23}$$

$$\sum_{v\le w} \eta^v(w) = 1, \tag{24}$$

$$\text{if } w \in W^J, \sum_{v\in W^J, v\le w} \eta^v_J(w) = 1. \tag{25}$$

Proof. Because of the Coxeter relations for the generators $s \in S$, there is a $Q(V)$-algebra homomorphism $ev : Q(V)[W] \to Q(V)$ defined by $ev(s) = 1$ for $\forall s \in S$. Hence, $ev(Y_w) = \sum_{v\le w} q(v,w)$ by equation (21), and $ev(Y_w) = A(w)$ by equation (22). Therefore we have equality (23). Dividing both sides of equality (23) by $A(w)$, we get the equality (24) by Lemma 4.1 (i). The equality (25) follows from (24) and the definition of $\eta^v_J(w)$ (11). □

5. Main Theorem.

In the same setup as in Section 3.2, for any $x, y \in W^J$, denote by $x \xrightarrow{\beta} y$ if $yW_J = s_\beta x W_J$ and $x < y$. Then we have the following formula.

Theorem 5.1 (Hook formula for Coxeter group)**.** *Let $\chi \in V$ be dominant with stabilizer subgroup $stab_W(\chi) = W_J$. For any $w \in W^J$, the following equality holds.*

$$\sum \frac{m_k}{m_1\beta_1 + m_2\beta_2 + \cdots + m_k\beta_k} \cdot \ldots \cdot \frac{m_1}{m_1\beta_1} = \prod_{\beta\in S(w)} \left(1 + \frac{1}{\beta}\right), \tag{26}$$

where the sum is over all directed paths

$$x_k \xrightarrow{\beta_k} x_{k-1} \xrightarrow{\beta_{k-1}} \ldots \xrightarrow{\beta_1} x_0 = w \text{ in } W^J, \tag{27}$$

for any integer $k \ge 0$, and $m_i := (\chi, \gamma_i^\vee)$ for the unique $\gamma_i \in R^+$ such that $x_{i-1}W_J = x_i s_{\gamma_i} W_J$ $(1 \le i \le k)$.

Taking the lowest degree terms in equation (26)*, we get*

$$\sum \frac{m_k}{m_1\beta_1 + m_2\beta_2 + \cdots + m_k\beta_k} \cdot \ldots \cdot \frac{m_1}{m_1\beta_1} = \prod_{\beta\in S(w)} \frac{1}{\beta}, \tag{28}$$

where the sum is over all directed sequences as in (27) *with length $k = \ell(w)$.*

Proof. By Lemma 2.2, $c^{w,J}_{\chi,w} - c^{u,J}_{\chi,u} = w\chi - u\chi \ne 0$ for any $u < w \in W^J$. Therefore we can apply the formula (17) recursively to get

$$d^{w,J}_{u,w} = \sum \left(\prod_{i=1}^{k} \frac{c^{x_i,J}_{\chi,x_{i-1}}}{c^{w,J}_{\chi,w} - c^{x_{i-1},J}_{\chi,x_{i-1}}}\right) d^{w,J}_{w,w}, \tag{29}$$

where the summation is over all integers $k \ge 1$, and sequences $u = x_0 < x_1 < x_2 < \cdots < x_k = w, x_i \in W^J$. On the other hand, by (25) and Lemma 3.3, $\sum_{u\le w} d^{w,J}_{u,w} = 1$ and we have equality $\sum_{u\in W^J, u\le w} \frac{d^{w,J}_{u,w}}{d^{w,J}_{w,w}} = \prod_{\beta\in S(w)} \frac{1+\beta}{\beta}$. Then the theorem follows from Lemma 2.3 and Lemma 3.2. □

An element $w \in W$ is said to be χ-minuscule $(\chi \in V)$ if

$$(s_{i_{j+1}} s_{i_{j+2}} \cdots s_{i_\ell}(\chi), \alpha_j^\vee) = 1 \ \ (j = 1, \ldots \ell)$$

for a reduced decomposition $w = s_{i_1}\cdots s_{i_\ell}$.

Lemma 5.2 ([8, Lemma 3.5, Corollary 3.10])**.** *In the setting of Theorem 5.1, if $u \in W$ is χ-minuscule, then $u \in W^J$ and $m_i = 1$ $(1 \le i \le k)$ for each directed path* (27)*.*

Corollary 5.1 (Nakada's colored hook formula [10, Theorem 7.1])**.** *Let W be the Weyl group of a Kac–Moody algebra, with the set of simple reflections S, acting on the real Cartan subalgebra $\mathfrak{h}_{\mathbf{R}}$. Let $\lambda \in V = \mathfrak{h}^*_{\mathbf{R}}$ be a dominant integral weight, $stab_W(\lambda) = W_J$, and $w \in W$ be a λ-minuscule element. Then we have*

(30)
$$\sum \frac{1}{\beta_1+\beta_2+\cdots+\beta_k}\cdot \cdots \cdot \frac{1}{\beta_1} = \prod_{\beta\in S(w)} \left(1+\frac{1}{\beta}\right),$$

where the sum is over all integers $k \geq 0$ and directed paths (27)*.*

Remark 5.1. (i) Nakada's original formula is written in terms of pre-dominant integral weights. The equivalence of the formulation above is explained in [8, Section 3.2].

(ii) The terminology 'hook' arises from the Grassmannian situation, where we may consider each $\beta \in S(w)$ as a hook in the partition for w.

(iii) There is a K-theoretic analog of Theorem. 5.1, which will be considered elsewhere.

6. Appendix: Geometric interpretation. Here we comment briefly on the geometric interpretation of our construction for the finite Weyl group case. In [8], we noticed that for a finite Weyl group W, Nakada's colored hook formula can be derived using geometric arguments via the Chern–Schwartz–MacPherson (CSM) classes of the Schubert cells. Let G be a reductive algebraic group with Borel B, maximal torus T, and Weyl group W. Let $X := G/B$ be the full flag variety, and $H_*^T(X)$, $H_T^*(X)$ denote the T-equivariant homology and T-equivariant cohomology, respectively. For any $w \in W$, there are Schubert cells $X(w)^\circ = BwB/B$, $Y(w)^\circ = B^- wB/B$, and their closures $X(w) = \overline{X(w)^\circ}$, $Y(w) = \overline{Y(w)^\circ}$ inside X. We refer the readers to [7] and [8] for unexplained terminology below.

There is a left Weyl group action on $H_*^T(G/B)$. For any $w \in W$, let us denote its action by δ_w. Then we have the following correspondence:

∂_w	Schubert class $[X(w)] \in H_*^T(X)$,
$\mathcal{L}_w$	CSM class of the Schubert cell $c_{SM}(X(w)^\circ) \in H_*^T(X)$,
η^v	Segre–MacPherson class of the opposite Schubert cell $s_M(Y(v)^\circ) \in H_T^*(X)_{loc}$,
$\eta^v(w)$	the localization $s_M(Y(v)^\circ)\|_w$.

To be more precise, the operator $\mathcal{L}_w$ becomes the left Demazure–Lusztig operator, denoted by $\mathcal{T}_w^L$ in [7], and it is shown in Theorem 4.4 of *loc. cit.* that

(31) $$\mathcal{T}_w^L(c_{SM}(X(id)^\circ)) = c_{SM}(X(w)^\circ).$$

The other identifications can be proved similarly.

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