133. *Probability-theoretic Investigations on Inheritance.* IV$_6$. *Mother-child Combinations.*

(Further Continuation.)

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(Comm. by T. FURUHATA, M.J.A., Nov. 12, 1951.)

5$^{\text{bis}}$. Mother-child-child combination.

Comparison between the corresponding probabilites of the mother-children combination and of the mother-child-child combination shows a remarkable fact. Namely, *with respe t to the corresponding probabilities laid on the principal diagonals in both tables, the value in the former table is generally greater than that in the latter.* This may be a fact previously expected as a reasonable matter. Indeed, since in the former case both children have a father also in common, it is natural that the tendency of resemblance between their types is stronger than in the latter case. Accordingly, *with respect to the probabilities not laid on the principal diagonals, the value in the former will be, on the contrary, less then the corresponding one in the latter, except a few;* furthermore the value in the former is, as soon shown, for the most part equal to a half of that in the latter. Hence, we may state that *few resembling types appear, in short, less in the former.*

Precise comparison with respect to individual separate probabilities will be made as follows. We denote by $\pi_0(ij;\,hk,fg)$ *the quantity obta'ned from* (5.6) *after the specialization* (5.3); *namely, we define*

$$\pi_0(ij\,;\,hk,fg) = [\pi^*(ij\,;\,hk,fg)]^{(p')=(p'')=(p)} \tag{5.9}$$

For brevity's sake, we further *introduce the notation*

$$d(ij\,;\,hk,fg) \equiv \pi(ij\,;\,hk,fg) - \pi_0(ij\,;\,hk,fg). \tag{5.10}$$

We now compare the corresponding probabilities laid on a principal diagonal. We first see that

$$d(ii\,;\,ii,ii) = \tfrac{1}{2}p_i^3(1+p_i) - p_i^4 = \tfrac{1}{2}p_i^3(1-p_i) \geqq 0\,, \tag{5.11}$$

$$d(ii\,;\,ih,ih) = \tfrac{1}{2}p_i^2p_h(1+p_i) - p_i^2p_h^2 = \tfrac{1}{2}p_i^2p_h(1-p_h) \geqq 0\,; \tag{5.12}$$

the corresponding result on the triple $(ii\,;\,ik,\,ik)$ can be regarded as to be contained essentially in (5.12), The equality sign of the inequality in (5.11) does never appear, unless a degenerate case

$p_i = 0$ or $p_i = 1$ occurs, and similarly, that in (5.12) can appear only if $p_i = 0$ or $p_h = 0$ or $p_h = 1$. We next see that

$$(5.13)\quad d(ij\,;ii,ii) = \tfrac{1}{4}p_i^2p_j(1+p_i)-\tfrac{1}{2}p_i^3p_j = \tfrac{1}{4}p_i^2p_j(1-p_i) \geqq 0\,,$$

$$(5.14)\quad \begin{aligned} d(ij\,;ij,ij) &= \tfrac{1}{4}p_ip_j(p_i+p_j)(1+p_i+p_j)-\tfrac{1}{2}p_ip_j(p_i+p_j)^2 \\ &= \tfrac{1}{4}p_ip_j(p_i+p_j)(1-p_i-p_j) \geqq 0\,, \end{aligned}$$

$$(5.15)\quad d(ij\,;ih,ih) = \tfrac{1}{4}p_ip_jp_h(1+p_h)-\tfrac{1}{2}p_ip_jp_h = \tfrac{1}{4}p_ip_jp_h(1-p_h) \geqq 0\,.$$

The comparison between the corresponding probabilities laid on the principal diagonals has thus essentially been worked out.

Comparison between the remaining corresponding probabilities takes place as follows:

$$(5.16)\quad \begin{cases} \pi(ii\,;ii,ih) = \tfrac{1}{2}\pi_0(ii\,;ii,ih), \\ \pi(ii\,;ih,ik) = \tfrac{1}{2}\pi_0(ii\,;ih,ik). \end{cases}$$

In case of heterozygotic mother, we get, in general,

$$(5.17)\quad \pi(ij\,;hk,fg) = \tfrac{1}{2}\pi_0(ij\,;hk,fg) \qquad (i \neq j),$$

except only cases where

$$(5.18)\quad \begin{cases} d(ij\,;ii,ij) = d(ij\,;ij,ii) = \tfrac{1}{4}p_i^2p_j(1+p_i+p_j)-\tfrac{1}{2}p_i^2p_j(p_i+p_j) \\ \qquad\qquad\qquad\qquad\quad = \tfrac{1}{4}p_i^2p_j(1-p_i-p_j) \geqq 0\,, \\ d(ij\,;jj,ij) = d(ij\,;ij,jj) = \tfrac{1}{4}p_ip_j^2(1-p_i-p_j) \geqq 0\,, \end{cases}$$

$$(5.19)\quad \begin{aligned} d(ij\,;ih,jh) = d(ij\,;jh,ih) &= \tfrac{1}{4}p_ip_jp_h(1+p_h)-\tfrac{1}{2}p_ip_jp_h^2 \\ &= \tfrac{1}{4}p_ip_jp_h(1-p_h) \geqq 0\,. \end{aligned}$$

The comparison between the corresponding probabilities not laid on the principal diagonals has also essentially been worked out.

Analogous relations are also observed on phenotypes. With respect to an inherited character in which no dominance relation is existent, such as *MN* blood type, the fact is immediately obvious. Suppose now that dominance relations are existent. We make use of the notation Π introduced in (3.27) of IV, define a similar notation Π_0 obtained from Π by substituting π_0's instead of π's, and then put

$$(5.20)\quad D(i\,;h,f) \equiv \Pi(i\,;h,f)-\Pi_0(i\,;h,f).$$

We now assert that the inequality

$$(5.21)\quad D(i\,;h,h) \geqq 0$$

is valid for every pair of i and h, being not necessarily different each other.

In order to prove the assertion, we first consider the case $h = i$, and obtain

$$
\begin{aligned}
D(i;i,i) &= \sum_{a=1}^{\alpha}\sum_{b=1}^{\alpha}\sum_{c=1}^{\alpha} d(ii_a;ii_b,ii_c) \qquad (i\equiv i_1)\\
&=\sum_{b=1}^{\alpha}\Big(d(ii;ii_b,ii_b)+\sum_{c\neq b} d(ii;ii_b,ii_c)\Big)\\
&+\sum_{a=2}^{\alpha}\Big(d(ii_a;ii,ii)+d(ii_a;ii,ii_a)+\sum_{c\neq 1,a} d(ii_a;ii,ii_c)\Big)\\
&+\sum_{a=2}^{\alpha}\Big(d(ii_a;ii_a,ii)+d(ii_a;ii_a,ii_a)+\sum_{c\neq 1,a} d(ii_a;ii_c,ii_a)\Big)\\
&+\sum_{a=2}^{\alpha}\sum_{b\neq 1,a}\Big(d(ii_a;ii_b,ii_b)+\sum_{c\neq b} d(ii_a;ii_b,ii_c)\Big)\\
\text{(5.22)}\quad &=\sum_{b=1}^{\alpha}\Big(\tfrac{1}{2}p_i^2 p_{i_b}(1-p_{i_b})-\sum_{c\neq b}\tfrac{1}{2}p_i^2 p_{i_b}p_{i_c}\Big)\\
&+\sum_{a=2}^{\alpha}\Big(\tfrac{1}{4}p_i^2 p_{i_a}(1-p_i)+\tfrac{1}{4}p_i^2 p_{i_a}(1-p_i-p_{i_a})-\sum_{c\neq 1,a}\tfrac{1}{4}p_i^2 p_{i_a}p_{i_c}\Big)\\
&+\sum_{a=2}^{\alpha}\Big(\tfrac{1}{4}p_i^2 p_{i_a}(1-p_i-p_{i_a})+\tfrac{1}{4}p_i p_{i_a}(p_i+p_{i_a})(1-p_i-p_{i_a})-\sum_{c\neq 1,a}\tfrac{1}{4}p_i p_{i_a}p_{i_c}(p_i+p_{i^a})\Big)\\
&+\sum_{a=2}^{\alpha}\sum_{b\neq 1,a}\Big(\tfrac{1}{4}p_i p_{i_a}p_{i_b}(1-p_{i_b})-\sum_{c\neq b}\tfrac{1}{4}p_i p_{i_a}p_{i_b}p_{i_c}\Big)\\
&=\tfrac{1}{2}p_i^2\sum_{b=1}^{\alpha}p_{i_b}^2\Big(1-\sum_{c=1}^{\alpha}p_{i_c}\Big)+\tfrac{1}{4}p_i^2\sum_{a=2}^{\alpha}p_{i_a}\Big(1-p_i+1-\sum_{c=1}^{\alpha}p_{i_c}\Big)\\
&+\tfrac{1}{4}p_i\sum_{a=2}^{\alpha}\Big(p_i p_{ia}(1-p_i-p_{i_a})+p_{i_a}(p_i+p_{i_a})\Big(1-\sum_{c=1}^{\alpha}p_{i_c}\Big)\Big)\\
&\qquad\qquad+\tfrac{1}{4}p_i\sum_{c=2}^{\alpha}p_{i_a}\sum_{b\neq 1,a}p_{i_b}\Big(1-\sum_{c=1}^{\alpha}p_{i_a}\Big);
\end{aligned}
$$

each term of the last expression and hence the expression itself is evidently non-negative.

We next consider the case $h \neq i$, more precisely stating, the case $h_1 \neq i_1$. Then, in the expression

$$
\text{(5.23)}\qquad D(i;h,h)=\sum_{a=1}^{\alpha}\sum_{b=1}^{\beta}\sum_{c=1}^{\beta} d(ii_a;\ hh_b,hh_c),
$$

since $d(ii_a;hh_b,hh_c)$ vanishes identically provided

$$
(i-h_b)(i_a-h)(i_a-h_b)\neq 0 \quad\text{or}\quad (i-h_c)(i_a-h)(i_a-h_c)\neq 0,
$$

it is sufficient to take only the remaining terms into account. *We now suppose,* as being really the case for all human blood types known at present, *that the dominance relations are subject to a semi-order condition*; namely, dominance of A_i against A_j and that of A_j against A_h imply always that of A_i against A_h.

Now, if the mother is of a type representing a dominant character, only the cases are really in question where a dominance relation exists between A_i and A_h, precisely stating, between A_{i_1}

and A_{h_1}. If the former is dominant against the latter we may assume that, in (5.23),

$$h_b = i_{\alpha-\beta+b} \qquad (b = 1, \dots, \beta), \tag{5.24}$$

and then obtain

$$\begin{aligned} D(i;\, h,\, h) &= \sum_{b=1}^{\beta}\Big(d(ih\,;\, hh_b, hh_b) + \sum_{c\neq b} d(ih\,;\, hh_b, hh_c)\Big) \\ &\qquad + \sum_{b=2}^{\beta} d(ih_b\,;\; hh_b, hh_b) \\ &= \sum_{b=1}^{b}\Big(\tfrac{1}{4}\, p_i p_h p_{h_b}(1-p_{h_b}) - \sum_{c\neq b}\tfrac{1}{4}\, p_i p_h p_{h_b} p_{h_c}\Big) \\ &\qquad + \sum_{b=2}^{\beta}\tfrac{1}{4}\, p_i p_h p_{h_b}(1-p_h) \\ &= \tfrac{1}{4} p_i p_h \sum_{b=1}^{\beta} p_{h_b}\Big(1-\sum_{c=1}^{\beta} p_{h_c}\Big) + \tfrac{1}{4}\, p_i p_h(1-p_h) \sum_{b=2}^{\beta} p_{h_b}\,; \end{aligned} \tag{5.25}$$

if the former is recessive against the latter, we may assume, instead of (5.24), that

$$i_a = h_{\beta-\alpha+a} \qquad (a = 1, \dots, \alpha), \tag{5.26}$$

and then obtain

$$\begin{aligned} D(i\,;\, h, h) &= d(ii\,;\, hi, hi) \\ &+ \sum_{a=2}^{\alpha}\Big(d(ii_a\,;\, hi, hi) + d(ii_a;\, hi, hi_a) + d(ii_a;\, hi_a, hi) + d(ii_a\,;\, hi_a, hi_a)\Big) \\ &= \tfrac{1}{2}\, p_i^2 p_h(1-p_h) + \sum_{a=2}^{\alpha}\Big(\tfrac{1}{4}\, p_i p_{i_a} p_h(1-p_h) + \tfrac{1}{4}\, p_i p_{i_a} p_h(1-p_h) \\ &+ \tfrac{1}{4}\, p_i p_{i_a} p_h(1 p_h) + \tfrac{1}{4}\, p_i p_{i_a} p_h(1-p_h)\Big) = \tfrac{1}{2}\, p_i p_h(1-p_h)\Big(p_i + 2\sum_{a=2}^{\alpha} p_{i_a}\Big). \end{aligned} \tag{5.27}$$

It is evidently seen that (5.25) and (5.27) are both always non-negative. Thus, our assertion has been performed.

We next assert that the analogous inequality

$$D(ij\,;\, h, h) \geqq 0 \tag{5.28}$$

is also valid for every triple i, j, h.

To prove this, we first consider the case $i = h \equiv h_1$ where necessariiy $j \neq h_b$ $(1 \leqq b \leqq \beta)$, and obtain

$$\begin{aligned} D(ij\,;\, h, h) \equiv D(hj\,;\, h, h) &= \sum_{b=1}^{\beta}\sum_{c=1}^{\beta} d(hj\,;\; hh_b, hh_c) \\ &= \sum_{b=1}^{\beta}\Big(d(hj\,;\, hh_b, hh_b) + \sum_{c\neq b} d(hj\,;\, hh_b, hh_c)\Big) \\ &= \sum_{b=1}^{\beta}\Big(\tfrac{1}{4}\, p_h p_j p_{h_b}(1-p_{h_b}) - \sum_{c\neq b}\tfrac{1}{4}\, p_h p_j p_{h_b} p_{h_c}\Big) \\ &= \tfrac{1}{4}\, p_h p_j \sum_{b=1}^{\beta} p_{h_b}\Big(1-\sum_{c=1}^{\beta} p_{h_c}\Big) \geqq 0. \end{aligned} \tag{5.29}$$

In case $i=h_b$ for a b with $2\leqq b\leqq\beta$, we obtain more simply

$$(5.30)\qquad D(h_bj\,;h,h)=d(h_bj\,;hh_b,hh_b)=\tfrac{1}{4}\,p_hp_jp_{h_b}(1-p_h)\geqq 0\,;$$

and quite similarly in case $j=h_b$ for a b with $2\leqq b\leqq\beta$. Thus, the inequality (5.28) is verified for every possible case.

We can deduce further analogous inequalities, stating that

$$(5.31)\qquad D(i\,;hk,hk)\geqq 0$$

and

$$(5.32)\qquad D(ij\,;hk,hk)\geqq 0,$$

valid for any set of i,j,h,k.

In fact, if $h=i$,

$$(5.33)\qquad \begin{aligned} D(i\,;ik,ik)&=\sum_{a=1}^{\alpha}d(ii_a\,;ik,ik)=d(ii\,;ik,ik)+\sum_{a=2}^{\alpha}d(ii_a\,;ik,ik)\\ &=\tfrac{1}{2}\,p_i^2p_k(1-p_k)+\sum_{a=2}^{\alpha}\tfrac{1}{4}\,p_ip_{i_a}p_k(1-p_k)\\ &=\tfrac{1}{4}\,p_ip_k(1-p_k)\Big(p_i+\sum_{a=1}^{\alpha}p_{i_a}\Big)\geqq 0\,; \end{aligned}$$

if $h=i_a$ for an a with $2\leqq a\leqq\alpha$,

$$(5.34)\qquad D(i\,;i_ak,i_ak)=d(ii_a\,;i_ak,i_ak)=\tfrac{1}{4}\,p_ip_{i_a}p_k(1-p_k)\geqq 0\,,$$

and similarly if $k=i_a$ for an a with $2\leqq a\leqq\alpha$.

With respect to (5.32) we have essentially to consider the case $i=h$ alone and then obtain

$$(5.35)\qquad D(ij\,;ik,ik)=d(ij\,;ik,ik)=\begin{cases}\tfrac{1}{4}\,p_ip_j(p_i+p_j)(1-p_i-p_j) & (k=j),\\ \tfrac{1}{4}\,p_ip_jp_k(1-p_k) & (k\neq j),\end{cases}$$

which is evidently non-negative. Thus, the inequalities (5.31) and (5.32) have both been proved.

We thus conclude the following fact: *If we compare the table of mother-children combination with that of mother-child-child combination, each concerning phenotypes, then every probability laid on the principal diagonals of the former is, in general, greater than the corresponding one of the latter, provided that the identically vanishing probabilities are excepted, each table being supposed to be constructed in such a manner that, for each type of the mother, the types of the first and the second children are arranged in the same order.*

Illustrative examples will be offered by *ABO* or *Q* blood type.

We have seen that the tendency of coicidence of types of both children is stronger in case of mother-children combination than in case of mother-child-child combination, in accordance with that, in the former case, both children have a father also in common while in the latter not. However, even in case of the latter, both children have, at any rate, a mother in common. Hence, it is reasonable to expect that the above mentioned tendency is still stronger in case of mother-child-child combination than in case of a pair of two children chosen at random. That it is really the case will be shown in the next chapter.

The discussions on probability a posteriori analogous to those at the ends in §§1–3 of IV apply also to that in the present section.

The results of the present section may be generalized to the several children case, corresponding to §4 of IV. The relation of the form

$$(5.36) \qquad \pi^*(ij\,;h_1k_1,\ldots,h_nk_n)=\frac{1}{\bar{A}_{ij}}\prod_{\nu=1}^{n}\pi^{(\nu)}(ij\,;h_\nu k_\nu)$$

will then play a fundamental role. However, beside this extreme case, there are various more general generalizations in intermediate stages. If the first n_1 children have a father in commen, the next n_2 have an another father in common, ..., the last n_α have a father different from the preceding ones in common, then the relation (5.36) must be replaced by that of the form

$$(5.37) \qquad \pi^*(ij\,;h_1k_1,\ldots,h_nk_n)=\frac{1}{\bar{A}_{ij}}\prod_{\nu=1}^{\alpha}\pi^{(\nu)}(ij\,;h_{\mu_{\nu-1}+1}k_{\mu_{\nu-1}+1},\ldots,h_{\mu_\nu}k_{\mu_\nu}),$$

where

$$\mu_0=0\,,\qquad \mu_\nu=n_1+\ldots+n_\nu\qquad(1\leqq\nu\leqq\alpha)$$

and

$$n=\mu_\alpha\equiv\sum_{\nu=1}^{\alpha}n_\nu\,.$$