131. *Probability-theoretic Investigations on Inheritance.* IV_4. *Mother-Child Combinations.*

(Further Continuation.)

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4. Mother-children combination concerning families with several children.

We have discussed, in the preceding section, the probabilities of mother-children combinations concerning families with two children. The results can be further generalized to several children case. We now consider the set of a mother and her n children produced from a common father, n being arbitrary but fixed.

Consider again an inherited character consisting of m genes A_i $(i=1,\ldots,m)$ with distribution-probability $\{p_i\}$, the distribution being here also supposed to be in an equilibrium state. In general, the number of permutations, admitting the repetition, of selecting any n types of children without kinship is equal to

$$(4.1)\qquad 2^{-n}m^n(m+1)^n.$$

But, if the children are restricted such that they have a common mother, then the corresponding number becomes

$$(4.2)\qquad m^n \quad\text{or}\quad (2m-1)^n$$

according to the mother of a homozygote or of a heterozygote, respectively. If they are further restricted such as to have a father also in common, then number of possible permutations reduces to a very small one. In fact, corresponding to that in §3 of IV, we get the following table.

Mating	$A_{ii}\times A_{ii}$	$A_{ii}\times A_{ik}$	$A_{ii}\times A_{hh}$	$A_{ii}\times A_{hk}$	$A_{ij}\times A_{ij}$	$A_{ij}\times A_{ik}$	$A_{ij}\times A_{hk}$
Permutation	1	2^n	1	2^n	3^n	4^n	4^n

Making use of a table on one-child case written in §3 of IV, we can easily construct the corresponding table on n-children case.

We denote by $\pi(A_{ij};\,A_{h_1k_1},\ldots,A_{h_nk_n})$ or briefly by

$$(4.3)\qquad \pi(ij;\,h_1k_1,\ldots,h_nk_n)\quad (i,j,h_\nu,k_\nu=1,\ldots,m;\ \nu=1,\ldots,n)$$

the probability of appearing of a combination of a mother A_{ij} and her n children among which νth child is of type $A_{h_\nu k_\nu}$ for $\nu=1,\ldots,n$. This quantity is, as before, equal to zero provided either of n relations holds:

$$(4.4) \qquad (i-h_\nu)(i-k_\nu)(j-h_\nu)(j-k_\nu) \neq 0 \qquad (\nu = 1, \dots, n).$$

In general, the different types of children belonging to the same family are at most 4; moreover, if the mother is of a homozygote, they reduce to at most two. Symmetry relations

$$(4.5) \quad \begin{aligned} &\pi(ij; \dots, h_\nu k_\nu, \dots) = \pi(ji; \dots, h_\nu k_\nu, \dots) = \pi(ij; \dots, k_\nu h_\nu, \dots) \\ &= \pi(ji; \dots, k_\nu h_\nu, \dots) \qquad (\nu=1, \dots, n) \end{aligned}$$

result immediately from the definition. Thus, we can make again an *agreement* corresponding to that immediately subsequent to (3.3) of IV. Symmetry relations

$$(4.6) \quad \pi(ij; \dots, \overset{\mu}{\breve{h_\mu k_\mu}}, \dots, \overset{\nu}{\breve{h_\nu k_\nu}}, \dots) = \pi(ij; \dots, \overset{\mu}{\breve{h_\nu k_\nu}}, \dots, \overset{\nu}{\breve{h_\mu k_\mu}}, \dots) \quad (\mu, \nu = 1, \dots, n)$$

corresponding to (3.4) of IV, are also obvious.

Now, a mother of homozygote A_{ii} can produce children of at most two types among m possible types containing the gene A_i. There are m cases where n children are together of the same type. All the cases where $n-\nu$ $(0 < \nu < n)$ children are of the same type each other and the remaining ν are of the same type each other but different from the former $n-\nu$, amount to

$$\binom{m}{2}\binom{n}{\nu} = \frac{m(m-1)}{2} \frac{n!}{\nu!\,(n-\nu)!}.$$

On the other hand, there exist m possible types of mother with homozygote. Hence, the whole number of non-vanishing combination-probabilities, containing homozygotic mothers, is equal to

$$(4.7) \qquad m\left(m + \binom{m}{2}\sum_{\nu=1}^{n-1}\binom{n}{\nu}\right) = m^2(1 + (2^{n-1}-1)(m-1)).$$

Next, a mother of heterozygote A_{ij} $(i \neq j)$ can produce children of at most four types among $2m-1$ possible types containing at least one of A_i and A_j. There are $2m-1$ cases where n children are together of the same type. The cases where $n-\nu$ $(0<\nu<n)$ are of the same type each other and the remaining ν are of the same type each other but different from the former $n-\nu$, amount to

$$\binom{2m-1}{2}\frac{n!}{\nu!\,(n-\nu)!}.$$

The cases where n children are divided into three classes of different types consisting of $n-\mu-\nu, \mu, \nu$ $(0 < \mu, \nu; \mu+\nu < n)$ children amount to

$$\binom{2m-1}{3}\frac{n!}{\mu!\,\nu!\,(n-\mu-\nu)!}$$

and those where n children are divided into four classes of different types consisting of $n-\lambda-\mu-\nu, \lambda, \mu, \nu$ $(0<\lambda, \mu, \nu; \lambda+\mu+\nu<n)$ children, to

$$\binom{2m-1}{4}\frac{n!}{\lambda!\,\mu!\,\nu!\,(n-\lambda-\mu-\nu)!}$$

In view of the identities

$$\sum_{\substack{0<\mu,\nu\\ \mu+\nu<n}}\frac{n!}{\mu!\,\nu!\,(n-\mu-\nu)!}=\sum_{\substack{0\leq\mu,\nu\\ \mu+\nu\leq n}}\frac{n!}{\mu!\,\nu!(n-\mu-\nu)!}-\sum_{\nu=0}^{n}\frac{n!}{\nu!\,(n-\nu)!}-2\sum_{\nu=1}^{n-1}\frac{n!}{\nu!\,(n-\nu)!}-1$$
$$=(1+1+1)^n-(1+1)^n-2(2^n-2)-1=3(3^{n-1}-2^n+1)$$

and

$$\sum_{\substack{0<\lambda,\mu,\nu\\ \lambda+\mu+\nu<n}}\frac{n!}{\lambda!\,\mu!\,\nu!(n-\lambda-\mu-\nu)!}=\sum_{\substack{0\leq\lambda,\mu,\nu\\ \lambda+\mu+\nu\leq n}}\frac{n!}{\lambda!\,\mu!\,\nu!\,(n-\lambda-\mu-\nu)!}-\sum_{\substack{0\leq\mu,\nu\\ \mu+\nu\leq n}}\frac{n!}{\mu!\,\nu!\,(n-\mu-\nu)!}$$
$$-3\sum_{\substack{0<\mu,\nu\\ \mu+\nu<n}}\frac{n!}{\mu!\,\nu!\,(n-\mu-\nu)!}-3\sum_{\nu=1}^{n-1}\frac{n!}{\nu!\,(n-\nu)!}-1$$
$$=(1+1+1+1)^n-(1+1+1)^n-3\times3(3^{n-1}-2^n+1)-3\times2(2^{n-1}-1)-1=4(4^{n-1}-3^n+3\cdot2^{n-1}-1),$$

we see that the number of non-vanishing combination-probabilities, containing heterozygotic mothers, is equal to

$$\binom{m}{2}\Bigl(2m-1+\binom{2m-1}{2}\sum_{\nu=1}^{n-1}\frac{n!}{\nu!\,(n-\nu)!}+\binom{2m-1}{3}\sum_{\substack{0<\mu,\nu\\ \mu+\nu<n}}\frac{n!}{\mu!\,\nu!\,(n-\mu-\nu)!}$$
$$(4.8)\qquad +\binom{2m-1}{4}\sum_{\substack{0<\lambda,\mu,\nu\\ \lambda+\mu+\nu<n}}\frac{n!}{\lambda!\,\mu!\,\nu!\,(n-\lambda-\mu-\nu)!}\Bigr)=\frac{1}{6}\,m(m-1)(2m-1)$$
$$\times(3+6(2^{n-1}-1)(m-1)+3(3^{n-1}-2^n+1)(m-1)(2m-3)$$
$$+2(4^{n-1}-3^n+3\cdot2^{n-1}-1)(m-1)(2m-3)(m-2)).$$

Thus, the total number of non-vanishing combination-probabilities reduces to the sum of (4.7) and (4.8), namely

$$m^2(2^{n-1}(m-1)-(m-2))+\tfrac{1}{6}m(m-1)(2m-1)(2\cdot4^{n-1}(m-1)$$
$$(4.9)\qquad \times(2m-3)(m-2)-3^n(m-1)(2m-3)(2m-5)+3\cdot2^n(m-1)$$
$$\times(m-2)(2m-5)-(m-2)(2m-3)(2m-5)).$$

The result for $n=2$ is a special case contained in (4.9).

If we remember the symmetry relations (4.6), the number of non-vanishing combination-probabilities essentially different each other will be extremely less than (4.9). Indeed, in the case of homozygotic mothers, such a number is at most

$$(4.10)\qquad m\Bigl(m+\binom{m}{2}\Bigr)=\frac{1}{2}m^2(m+1).$$

In case of heterozygotic mothers, three cases are distinguished according to $n=2$, $n=3$, $n\geqq4$, and such a number is at most

$$\binom{m}{2}\Bigl(2m-1+\binom{2m-1}{2}\Bigr)=\frac{1}{2}m^2(m-1)(2m-1)\qquad(n=2),$$

$$\binom{m}{2}\Bigl(2m-1+\binom{2m-1}{2}+\binom{2m-1}{3}\Bigr)$$
$$(4.11)\qquad =\tfrac{1}{6}m(m-1)(2m-1)(2m^2-2m+3)\qquad(n=3),$$

$$\binom{m}{2}\left(2m-1+\binom{2m-1}{2}+\binom{2m-1}{3}+\binom{2m-1}{4}\right)$$
$$=\tfrac{1}{12}m^2(m-1)(2m-1)(2m^2-5m+9) \qquad (n\geqq 4).$$

Hence, *the total number of non-vanishing combination-probabilities essentially different each other is*, as the sum of (4.10) and (4.11), *given by*

$$(4.12)\quad \begin{aligned} &m^2(m^2-m+1) && (n=2),\\ &\tfrac{1}{2}m^2(m+1)+\tfrac{1}{6}m(m-1)(2m-1)(2m^2-2m+3) && (n=3),\\ &\tfrac{1}{2}m^2(m+1)+\tfrac{1}{12}m^2(m-1)(2m-1)(2m^2-5m+9) && (n\geqq 4). \end{aligned}$$

The result for $n=2$ contained in (4.12) is nothing but the one already stated. It may be especially noticed that the number is independent of n provided $n\geqq 4$. For instance, if m is equal to 2, 3, 4 or 10, then the numbers in (4.12) become 13, 93, 418 or 52705 for $n=3$, and 13, 108, 628 or 227125 for $n\geqq 4$, respectively.

We now enter into our main discourse. The order of n children being indifferent, as remarked in (4.6), we denote briefly by

$$(4.13)\qquad \pi(ij\,;\,h_1k_1^{n_1},\,h_2k_2^{n_2},\,\ldots,\,h_\alpha k_\alpha^{n_\alpha}) \qquad \left(\sum_{\nu=1}^{\alpha} n_\nu = n\right)$$

the probability of the combination composed of a mother of A_{ij} *and the* 1*st to* n_1*th children of* $A_{h_1k_1}$, *the* (n_1+1)*th to* (n_1+n_2)*th children of* $A_{h_2k_2}$, ..., *the* $(n_1+\ldots+n_{\alpha-1}+1)$*th to* $(n_1+\ldots+n_{\alpha-1}+n_\alpha)=n$*th children of* $A_{h_\alpha k_\alpha}$, *where* α *is a number such as* $1\leqq\alpha\leqq 4$. By permutating the order of children, there will appear $n!\,/\,\Pi_{\nu=1}^{\alpha} n_\nu!$ probabilities which have the same value.

We first consider a mother of homozygote A_{ii}. Possible type of a father who can produce a child A_{ii} with this mother are A_{ii}, A_{ih} $(h\neq i)$. If the type of father is A_{ii}, then that of a child is always A_{ii}, while if the type of father is A_{ih}, then a child A_{ii} is produced in probability 1/2. Thus we get, as in (3.10) of IV,

$$(4.14)\qquad \pi(ii\,;\,ii^n)=1p_i^4+2^{-n}\,2p_i^3\sum_{h\neq i}p_h=2^{-n+1}\,p_i^3(1+(2^{n-1}-1)p_i).$$

The type of a father who can produce at least two children A_{ii} and A_{ih} $(h\neq i)$ with a mother A_{ii} must be A_{ih}, whence it follows

$$(4.15)\qquad \pi(ii\,;\,ii^{n-\nu},\,ih^\nu)=2^{-n}\,2p_i^3p_h=2^{-n+1}\,p_i^3p_h \qquad (h\neq i\,;\,0<\nu<n).$$

Possible types of a father who can produce a child $A_{ih}(h\neq i)$ are A_{ih}, A_{hh} and A_{hk} $(k\neq i,h)$, and hence we get

$$(4.16)\qquad \begin{aligned}\pi(ii\,;\,ih^n)&=2^{-n}\,2p_i^3p_h+1p_i^2p_h^2+2^{-n}\,2p_i^2p_h\sum_{k\neq i,h}p_k\\ &=2^{-n+1}\,p_i^2p_h\,(1+(2^{n-1}-1)p_h) \qquad (h\neq i).\end{aligned}$$

Since the only possible type of a father who can produce at least two children A_{ih} $(h\neq i)$ and A_{ik} $(k\neq i,h)$ is A_{hk}, we obtain

$$(4.17)\quad \pi(ii\,;\,ih^{n-\nu},ik^{\nu})=2^{-n}2p_i^2p_hp_k=2^{-n+1}\,p_i^2p_hp_k \qquad (h,k\neq i\,;\,h\neq k\,;\,0<\nu<n).$$

The last result remains valid, as seen from (4.15), if only one of h or k coincides with i. Thus, the case of a homozygotic mother has been worked out essentially.

We next consider a mother of heterozygote A_{ij} $(i\neq j)$. In case where n children are together of the same homozygote, we get

$$(4.18)\quad \begin{aligned}\pi(ij\,;\,ii^n)&=2^{-n}2p_i^3p_j+4^{-n}4p_i^2p_j^2+4^{-n}4p_i^2p_j\sum_{h\neq i,j}p_h\\&=4^{-n+1}\,p_i^2p_j(1+(2^{n-1}-1)p_i),\end{aligned}$$

$$(4.19)\quad \pi(ij;\,jj^n)=4^{-n+1}\,p_ip_j^2\,(1+(2^{n-1}-1)p_j)\,.$$

In case where n children are together of the same heterozygote, we get

$$(4.20)\quad \begin{aligned}\pi(ij;\,ij^n)&=2^{-n}\,2p_i^3p_j+2^{-n}2p_ip_j^3+2^{-n}4p_i^2p_j^2+4^{-n}\,4p_i^2p_j\sum_{h\neq i,j}p_h\\&+4^{-n}\,4p_ip_j^2\sum_{h\neq i,j}p_h=4^{-n+1}\,p_ip_j(p_i+p_j)(1+(2^{n-1}-1)(p_i+p_j)\,)\,,\end{aligned}$$

$$(4.21)\quad \begin{aligned}\pi(ij\,;\,ih^n)&=4^{-n}4p_i^2p_jp_h+4^{-n}4p_ip_j^2p_h+2^{-n}p_ip_jp_h^2\\&+4^{-n}4p_ip_jp_h\sum_{k\neq i,j,h}p_k=4^{-n+1}\,p_ip_jp_h(1+(2^{n-1}-1)p_h)\qquad (h\neq i,j),\end{aligned}$$

$$(4.22)\quad \pi(ij\,;\,jh^n)=4^{-n+1}\,p_ip_jp_h(1+(2^{n-1}-1)p_h)\qquad (h\neq i,j).$$

Only possible type of a father who can produce at least two children A_{ii} and A_{jj} is A_{ij}, whence it follows

$$(4.23)\quad \pi(ij\,;\,ii^{n-\nu},\,jj^{\nu})=4^{-n}\,4p_i^2p_j^2=4^{-n+1}\,p_i^2p_j^2\qquad (0<\nu<n).$$

Possible types of a father who can produce at least two children A_{ii} and A_{ij} are A_{ii}, A_{ij}, A_{ih} $(h\neq i,j)$, and hence we obtain

$$(4.24)\quad \begin{aligned}\pi(ij;\,ii^{n-\nu},ij^{\nu})&=2^{-n}\,2p_i^3p_j+4^{-n+\nu}2^{-\nu}\,4p_i^2p_j^2+4^{-n}\,4p_i^2p_j\sum_{h\neq i,j}p_h\\&=4^{-n+1}\,p_i^2p_j\,(1+(2^{n-1}-1)p_i+(2^{\nu}-1)p_j)\qquad (0<\nu<n),\end{aligned}$$

and similarly

$$(4.25)\quad \pi(ij\,;\,jj^{n-\nu},ij^{\nu})=4^{-n+1}\,p_ip_j^2\,(1+(2^{\nu}-1)p_i+(2^{n-1}-1)p_j) \qquad (0<\nu<n).$$

The last two results remain valid, as seen from (4.18) and (4.19), also for $\nu=0$.

In similar manners, we get in turn the following results:

$$(4.26)\quad \begin{aligned}\pi(ij\,;\,ih^n)&=4^{-n}\,4p_i^2p_jp_h+4^{-n}\,4p_ip_j^2p_h+2^{-n}\,2p_ip_jp_h^2\\&+4^{-n}4p_ip_jp_h\sum_{k\neq i,j,h}p_k=4^{-n+1}\,p_ip_jp_h(1+(2^{n-1}-1)p_h)\quad (h\neq i,j;\,0<\nu<n),\end{aligned}$$

$$(4.27)\quad \pi(ij;\,ii^{n-\mu-\nu},\,ih^{\mu},jh^{\nu})=4^{-n}4p_i^2p_jp_h=4^{-n+1}p_i^2p_jp_h\ (h\neq i,j;\,0<\mu+\nu<n),$$

$$(4.28)\quad \pi(ij\,;\,jj^{n-\mu-\nu},\,ih^{\mu},jh^{\nu})=4^{-n+1}p_ip_j^2p_h\qquad (h\neq i,j;\,0<\mu+\nu<n),$$

$$(4.29)\quad \begin{aligned}\pi(ij\,;\,ij^{n-\mu-\nu},\,ih^{\mu},jh^{\nu})&=4^{-n}4p_i^2p_jp_h+4^{-n}4p_ip_j^2p_h\\&=4^{-n+1}p_ip_jp_h(p_i+p_j)\qquad (h\neq i,j;\,0<\mu+\nu<n),\end{aligned}$$

$$(4.30)\quad \pi(ij;\,ii^{n-\mu-\nu},\,jj^{\mu},\,ij^{\nu})=4^{-n+\mu+\nu}4^{-\mu}2^{-\nu}4p_i^2p_j^2=2^{-2n+\nu+2}\,p_i^2p_j^2 \qquad (0<\mu\leq\mu+\nu<n),$$

(4.31) $\pi(ij; ii^{n-\lambda-\mu-\nu}, ij^{\lambda}, ih^{\mu}, jh^{\nu}) = 4^{-n}4p_i^2p_jp_h = 4^{-n+1}p_i^2p_jp_h$
$(h \neq i, j;\ 0 < \mu+\nu \leqq \lambda+\mu+\nu < n),$

(4.32) $\pi(ij; jj^{n-\lambda-\mu-\nu}, ij^{\lambda}, ih^{\mu}, jh^{\nu}) = 4^{-n+1}\, p_ip_j^2p_h$
$(h \neq i, j;\ 0 < \mu+\nu \leqq \lambda+\mu+\nu < n),$

(4.33) $\pi(ij; ih^{n-\lambda-\mu-\nu}, ik^{\lambda}, jh^{\mu}, jk^{\nu}) = 4^{-n}4p_ip_jp_hp_k = 4^{-n+1}p_ip_jp_hp_k$
$(h, k \neq i, j;\ h \neq k;\ 0 < \lambda+\nu < n).$

Thus, the case of a heterozygotic mother has also been worked out essentially.

It will be noticed that, for instance, the result (4.23) is contained in (4.30), as a special case $\nu = 0$. Moreover, in (4.31) to (4.33) the case $0 < \lambda, \mu, \nu < n$ does not really appear for $n < 4$.

The passage to the corresponding results on phenotypes in which recessive genes are interested can be done by means of the usual procedure. Finally, we notice that the mixed mother-child combinations could also be discussed; the fundamental interrelations analogous to (3.29) of IV being then of importance.

— To be continued —