46. On the Boundary Value of a bounded analytic Function of several complex Variables.

By Masatsugu TSUJI.
Mathematical Institute, Tokyo University.
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1. Let $f(z)$ be regular and bounded in $|z|<1$. Then (i) (Fatou.)[1] $\lim f(z)=f(e^{i\theta})$ exists almost everywhere on $|z|=1$, when z tends to $e^{i\theta}$ non-tangentially to $|z|=1$. (ii) (F. and M. Riesz.)[2] If the boundary value $f(e^{i\theta})$ vanishes on a set of positive measure on $|z|=1$, then $f(z)\equiv 0$. (iii) (Szegö.)[3] If $f(z)\not\equiv 0$, then $\log|f(e^{i\theta})|$ is integrable on $|z|=1$.

We will show that an analogous theorem holds for a bounded regular function of several complex-variables.

Let $z=e^{i\theta}$, $w=e^{i\varphi}$ be points on $|z|=1$, $|w|=1$ respectively. Then the pair $(e^{i\theta}, e^{i\varphi})$ can be considered as a point on a torus Θ $(0\leqq\theta\leqq 2\pi,\ 0\leqq\varphi\leqq 2\pi)$ and the measure of a measurable set E on Θ is defined by

$$mE=\iint_E d\theta d\varphi, \quad \text{so that } m\Theta=4\pi^2. \tag{1}$$

Then the following theorem holds:

Theorem 1. *Let $f(z, w)$ be regular and bounded in $|z|<1$, $|w|<1$. Then* (i) $\lim f(z, w)=f(e^{i\theta}, e^{i\varphi})$ *exists almost everywhere on Θ, when $z\to e^{i\theta}$, $w\to e^{i\varphi}$ non-tangentially to $|z|=1, |w|=1$ respectively.* (ii) *If the boundary value $f(e^{i\theta}, e^{i\varphi})$ vanishes on a set of positive measure on Θ, then $f(z, w)\equiv 0$.* (iii) *If $f(z, w)\not\equiv 0$, then $\log|f(e^{i\theta}, e^{i\varphi})|$ is integrable on Θ.*

Since I have proved (i) in the former paper,[4] I will prove (ii) and (iii). We remark that if $f(z, w)$ is bounded in $|z|<1$, $|w|<1$ and $|f(e^{i\theta}, e^{i\varphi})|\leqq M$ almost everywhere on Θ, then $|f(z, w)|\leqq M$ in $|z|<1$, $|w|<1$.

For, let $|z|<R<1$, $|w|<R<1$, then

$$f(z, w)=f(re^{i\theta}, \rho e^{i\varphi})$$
$$=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{f(Re^{i\theta'}, Re^{i\varphi'})(R^2-r^2)(R^2-\rho^2)d\theta' d\varphi'}{(R^2-2Rr\cos(\theta'-\theta)+r^2)(R^2-2R\rho\cos(\varphi'-\varphi)+\rho^2)}$$
$$(0\leqq r<R,\ 0\leqq\rho<R) \tag{2}$$

(1) P. Fatou: Séries trigonométriques et séries de Taylor, Acta Math. **30** (1906).

(2) F. und M. Riesz: Über die Randwerte einer analytischen Funktion. Compte rendu du quatrième congres des mathematiciens scandinaves (1916).

(3) G. Szegö: Über die Randwerte einer analytischen Funktion. Math. Ann. **84** (1921),

(4) M. Tsuji: On Hopf's ergodic theorem. Jap. Journ. Math. **19**.

so that for $R\to 1$, we have by Lebesgue's theorem,

$$f(z, w)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{f(e^{i\theta'},e^{i\varphi'})(1-r^2)(1-\rho^2)d\theta' d\varphi'}{(1-2r\cos(\theta'-\theta)+r^2)(1-2\rho\cos(\varphi'-\varphi)+\rho^2)},$$

hence

$$|f(z, w)|\leqq\frac{M}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{(1-r^2)(1-\rho^2)d\theta' d\varphi'}{(1-2r\cos(\theta'-\theta)+r^2)(1-2\rho\cos(\varphi'-\varphi)+\rho^2)}=M.$$

2. *Proof of* (ii).

Suppose that $|f(z, w)|\leqq M$ for $|z|<1$, $|w|<1$ and $|f(e^{i\theta}, e^{i\varphi})|\leqq m(<M)$ on a set E of positive measure on Θ. Then by Egoroff's theorem, there exists a closed sub-set E_0 of E, such that $mE_0\geqq mE-\epsilon>0$ and $\lim_{r\to 1} f(re^{i\theta}, re^{i\varphi}) =f(e^{i\theta}, e^{i\varphi})$ uniformly on E_0, so that there exists a suitable $R<1$, such that

$$|f(Re^{i\theta}, Re^{i\varphi})|\leqq m+\epsilon<M \quad \text{for} \quad (\theta, \varphi)\in E_0. \tag{3}$$

We put

$$F(\theta, \varphi)=\text{Max.}(\log(m+\epsilon), \log|f(Re^{i\theta}, Re^{i\varphi}))|). \tag{4}$$

Then $F(\theta, \rho)$ is continuous and

$$\begin{aligned} &F(\theta, \varphi)\leqq\log M \text{ on } \Theta-E_0,\\ &F(\theta, \varphi)=\log(m+\epsilon) \text{ on } E_0. \end{aligned} \tag{5}$$

Let for $|z|<R$, $|w|<R$,

$$\begin{aligned} u(z, w)&=u(re^{i\theta}, \rho e^{i\rho})\\ &=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{F(\theta', \varphi')(R^2-r^2)(R^2-\rho^2)d\theta' d\varphi'}{(R^2-2R\cos(\theta'-\theta)+r^2)(R^2-2R\rho\cos(\varphi'-\varphi)+\rho^2)}. \end{aligned}$$

We have

$$u(Re^{i\theta}, Re^{i\varphi})=F(\theta, \varphi)\geqq\log|f(Re^{i\theta}, Re^{i\varphi})|, \tag{6}$$

$$\begin{aligned} u(0, 0)&=\frac{1}{4\pi^2}\iint_{\Theta}F(\theta, \varphi)d\theta d\varphi=\frac{1}{4\pi^2}\iint_{E_0}F(\theta, \varphi)d\theta d\varphi\\ &+\frac{1}{4\pi^2}\iint_{\Theta-E_0}F(\theta, \varphi)\, d\theta d\varphi\leqq\log(m+\epsilon)\cdot\frac{mE_0}{4\pi^2}\\ &+\log M.\left(1-\frac{mE_0}{4\pi^2}\right). \end{aligned} \tag{7}$$

Let

$$\Phi(z, w)=\log|f(z, w)|-u(z, w), \; (|z|\leqq R, |w|\leqq R), \tag{8}$$

$$\mu=\underset{|z|\leqq R\ |w|\leqq R}{\text{upper limit}}\ \Phi(z, w). \tag{9}$$

Since $u(z, w)$ is bounded and $\log|f(z, w))|\leqq\log M$, we have $\mu<\infty$.

We will prove that $\mu\leqq 0$.

Let (z_n, w_n) be points in $|z|\leqq R$, $|w|\leqq R$, such that $z_n\to z_0$, $w_n\to w_0$ and

$\Phi(z_n, w_n)\to\mu$. Then $|z_0|\leqq R$, $|w_0|\leqq R$ and $\Phi(z_0, w_0)=\mu$, so that $f(z_0, w_0)\neq 0$.
(i) If $|z_0|<R$, $|w_0|<R$, then $f(z, w)\neq 0$ in $|z-z_0|\leqq\delta$, $|w-w_0|\leqq\delta$, where δ is taken so small that $|z-z_0|\leqq\delta$, $|w-w_0|\leqq\delta$ are contained in $|z|<R$, $|w|<R$ respectively. Then $\Phi(z, w)$ is a harmonic function of z and w in $|z-z_0|\leqq\delta$, $|w-w_0\leqq\delta$, so that if $\Phi(z, w)\not\equiv$ const., then $\Phi(z, w)$ takes values in $|z-z_0|\leqq\delta$, $|w-w_0|\leqq\delta$, which are greater than $\Phi(z_0, w_0)=\mu$, which contadicts the definition of μ. Hence $\Phi(z, w)\equiv\text{const.}=\mu$. Since by (6), (8), $\Phi(Re^{i\theta}, Re^{i\varphi})\leqq 0$, we have $\mu\leqq 0$.
(ii) Next suppose that $|z_0|<R$, $|w_0|=R$. We write $u(z, w)$ in the form:

$$u(z, w)=\frac{1}{2\pi}\int_0^{2\pi}\frac{U(\theta', w)(R^2-r^2)d\theta'}{R^2-2Rr\cos(\theta'-\theta)+r^2},$$

where

$$U(\theta, w)=\frac{1}{2\pi}\int_0^{2\pi}\frac{F(\theta, \varphi')(R^2-\rho^2)d\varphi'}{R^2-2R\rho\cos(\varphi'-\varphi)+\rho^2}.$$

Since $U(\theta, w)\to F(\theta, \varphi_0)$ for $w\to w_0=Re^{i\varphi_0}$,

$$u(z, w_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{F(\theta', \varphi_0)(R^2-r^2)d\theta'}{R^2-2Rr\cos(\theta-\theta)+r^2}.$$

Hence $u(z, w_0)$ is a harmonic function of z in $|z|<\text{R}$. Since $f(z_0, w_0)\neq 0$, $f(z,w_0)\neq 0$ in $|z-z_0|\leqq\delta$, where δ is taken so small that $|z-z_0|\leqq\delta$ is contained in $|z|<\text{R}$. Since $\Phi(z, w_0)$ is harmonic in $|z-z_0|\leqq\delta$, if $\Phi(z, w_0)\not\equiv$ const., then $\Phi(z, w_0)$ takes values in $|z-z_0|\leqq\delta$, which are greater than $\Phi(z_0, w_0)=\mu$, which contradicts the definition of μ. Hence $\Phi(z, w_0)\equiv\text{const.}=\mu$. Since $\Phi(Re^{i\theta}, R^{i\varphi})\leqq 0$, we have $\mu\leqq 0$.

(iii) We have $\mu\leqq 0$, if $|z_0|=R$, $|w_0|=R$ from (6), (8).

Hence $\mu\leqq 0$ in any case, so that

$$\log|f(z, w)|\leqq u(z, w) \text{ in } |z|\leqq R, |w|\leqq R,$$

hence by (7),

$$\log|f(0, 0)|\leqq u(0, 0)\leqq \log(m+\epsilon)\frac{mE_0}{4\pi^2}+\log M\left(1-\frac{mE_0}{4\pi^2}\right). \quad (10)$$

Making $\epsilon\to 0$, we have

$$\log|f(0, 0)|\leqq\log m\cdot\frac{mE}{4\pi^2}+\log M\cdot\left(1-\frac{mE}{4\pi^2}\right),$$

or

$$|f(0, 0)|\leqq m^{\frac{mE}{4\pi^2}}M^{1-\frac{mE}{4\pi^2}}. \quad (11)$$

Hence if $f(e^{i\theta}, e^{i\varphi})=0$ on a set E of positive measure on Θ, then $f(0, 0)=0$. From this we conclude that $f(z, w)\equiv 0$. For, if $f(z, w)\not\equiv 0$, let $f(z_0, w_0)\neq 0$

$(|z_0|<1, |w_0|<1)$. Put $F(z, w)=f\left(\frac{z+z_0}{1+\bar{z}_0 z}, \frac{w+w_0}{1+\bar{w}_0 w}\right)$. Then $F(z, w)$ is regular and bounded in $|z|<1, |w|<1$ and its boundary value vanishes on a set of positive measure on Θ, so that $F(0, 0)=f(z_0, w_0)=0$, which contradicts the hypothesis. Hence $f(z, w)\equiv 0$, q.e.d.

3. Proof of (iii).

Let $f(z, w)\not\equiv 0$, then after a suitable linear transformation, we may assume that $f(0, 0)\neq 0$ and $|f(z, w)|<1$ in $|z|<1, |w|<1$.

Let $|f(e^{i\theta}, e^{i\varphi})|\geqq 2\epsilon>0$ on a set E. Then by Egoroff's theorem, there exists a closed sub-set E_0 of E, such that $mE_0\geqq mE-\epsilon>0$ and $\lim_{r\to 1} f(re^{i\theta}, re^{i\varphi}) =f(e^{i\theta}, e^{i\varphi})$ uniformly on E_0, so that for a suitable $R<1$,

$$|f(Re^{i\theta}, Re^{i\varphi})|\geqq\epsilon \quad \text{for} \quad (\theta, \varphi)\in E_0. \tag{12}$$

Let

$$F(\theta, \varphi)=\text{Max.}(\log\epsilon, \log|f(Re^{i\theta}, Re^{i\varphi})|) \quad (0<\epsilon<1), \tag{13}$$

then $F(\theta, \varphi)$ is continuous and since $|f(z, w)|<1$, $0<\epsilon<1$,

$$F(\theta, \varphi)\leqq 0 \text{ on } \Theta-E_0, \quad F(\theta, \varphi)=\log|f(Re^{i\theta}, Re^{i\varphi})| \text{ on } E_0. \tag{14}$$

Let

$$u(z, w)=u(re^{i\theta}, \rho e^{i\theta})=$$

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{F(\theta', \varphi')(R^2-r^2)(R^2-\rho^2)d\theta' d\varphi'}{(R^2-2Rr\cos(\theta'-\theta)+r^2)(R^2-2R\rho\cos(\varphi'-\varphi)+\rho^2)}.$$

$$(0\leqq r<r<1, \ 0\leqq\rho<R<1)$$

Then by (10), (14),

$$\log|f(0, 0)|\leqq u(0, 0)=\frac{1}{4\pi^2}\iint_{\Theta-E_0}F(\theta, \varphi)d\theta d\varphi+\frac{1}{4\pi^2}\iint_{E_0}F(\theta, \varphi)d\theta d\varphi\leqq$$

$$\frac{1}{4\pi^2}\iint_{E_0}\log|f(Re^{i\theta}, Re^{i\varphi})|\,d\theta d\varphi.$$

Since for $R\to 1$, $\log|f(Re^{i\theta}, Re^{i\varphi})|\to\log|f(e^{i\theta}, e^{i\varphi})|$ uniformly on E_0,

$$\log|f(0, 0)|\leqq\frac{1}{4\pi^2}\iint_{E_0}\log|f(e^{i\theta}, e^{i\varphi})|\,d\theta d\varphi,$$

By (ii), $mE\to 4\pi^2$ for $\epsilon\to 0$, so that making $\epsilon\to 0$, we have

$$\log|f(0, 0)|\leqq\frac{1}{4\pi^2}\iint_{\Theta}\log|f(e^{i\theta}, e^{i\varphi})|\,d\theta d\varphi. \tag{15}$$

Hence $\log|f(e^{i\theta}, e^{i\varphi})|$ is integrable on Θ, q.e.d.

Similarly we can prove:

Theorem 2. Let $f(z_1, \dots, z_n)$ *be regular and bounded in* $|z_1|<1, \dots, |z_n|<1$. *Then* (i) $\lim f(z_1, \dots, z_n)=f(e^{i\theta_1}, \dots, e^{i\theta_n})$ *exists almost everywhere on an n-dimensional torus* $\Theta(0\leqq\theta_k\leqq 2\pi, k=1, 2, \dots, n)$, *when* $z_k\to e^{i\theta_k}$ *non-*

tangentially to $|z_k| = 1$ *respectively.* (ii) *If the boundary value* $f(e^{i\theta_1}, \dots, e^{i\theta_n})$ *vanishes on a set of positive measure on* $\boldsymbol{\Theta}$, *then* $f(z_1, \dots, z_n) \equiv 0$. (iii) *If* $f(z_1, \dots, z_n) \not\equiv 0$, *then* $\log|f(e^{i\theta_1}, \dots, e^{i\theta_n})|$ *is intgrable on* $\boldsymbol{\Theta}$.