37. Markoff Process and the Dirichlet Problem.

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(Comm. by T. TAKAGI, M.I.A., April 26, 1945)

1. The purpose of this paper is to give a general discussion of the Dirichlet problem from the standpoint of the theory of positive linear operations in a semi-ordered Banach space. It will be shown that the so-called sweeping out process of obtaining the solution of the Dirichlet problem may be observed as a kind of Markoff process[1] in the space of continuous functions.

2. Let Ω be a compact Hausdorff space. The set $C(\Omega)$ of all real-valued continuous functions $x(\omega)$ defined on Ω is a Banach space with respect to the norm:

(1) $$\|x\|=\sup_{\omega\in\Omega}|x(\omega)|.$$

$C(\Omega)$ is also an (M)-space[2] with respect to the partial ordering:

(2) $x\geqq y$ *if and only if* $x(\omega)\geqq y(\omega)$ *for all* $\omega\in\Omega$;

and $e(\omega)\equiv 1$ is the unit element of $C(\Omega)$.

3. Let D be a bounded domain in the Gaussian plane. We do not assume that D is simplv or finitely connected. Let us consider the (M)-spaces $C(\overline{D})$ and $C(\Gamma)$, where $\overline{D}$ is the closure of D and $\Gamma=\overline{D}-D$ is the boundary of D. For any $x(\zeta)\in C(\overline{D})$, let $y(\zeta)\in C(\Gamma)$ be the boundary value of $x(\zeta)$ on Γ. Then $y=A(x)$ is a bounded linear operation which maps $C(\overline{D})$ onto $C(\Gamma)$, and clearly satisfies

(3) $x\geqq 0$ *implies* $A(x)\geqq 0$,

(4) $x\equiv 1$ *implies* $A(x)\equiv 1$,

(5) $\|A(x)\|\leqq\|x\|$.

That $y=A(x)$ is an onto-mapping means the fact that, for any $y(\zeta)\in C(\Gamma)$, there exists an $x(\zeta)\in C(\overline{D})$ such that $A(x)=y$. We can take as $x(\zeta)$ any continuous extension of $y(\zeta)$ from Γ to $\overline{D}$. Such an extension, however, is not uniquely determined; but it is possible[3] to find in a concrete way a bounded linear operation $x=B(y)$ which maps $C(\Gamma)$ into $C(\overline{D})$ such that $AB(y)=y$ on $C(\Gamma)$ and further that

1) K. Yosida and S. Kakutani, Operator-theoretical treatment of Markoff process and the mean ergodic theorem, Annals of Math., 42(1941).

2. S. Kakutani, Concrete representation of abstract (M)-spaces and the characterization of the space of continuous functions, Annals of Math., 42(1941).

3) S. Kakutani, Simultaneous extension of continuous functions considered as a positive operation, Jap. Journ. of Math., 19(1940).

(6) $\quad y \geqq 0$ *implies* $B(y) \geqq 0,$

(7) $\quad y \equiv 1$ *implies* $B(y) \equiv 1,$

(8) $\quad \| B(y) \| = \| y \|.$

4. Let now D be a bounded domain in the Gaussian plane which is regular for the Dirichlet problem. Let $H(\overline{D})$ be the closed linear subspace of $C(\overline{D})$ consisting of all $x(\zeta) \in C(\overline{D})$ which are harmonic in D. That D is regular means that, for any $y(\zeta) \in C(\Gamma)$, there exists a $u(\zeta) \in H(\overline{D})$ such that $A(u) = y$. Such a $u(\zeta) \in H(\overline{D})$ is uniquely determined by $y(\zeta) \in C(\Gamma)$, and $u = U(y)$ thus defined is a bounded linear operation which maps $C(\Gamma)$ onto $H(\overline{D})$. It is clear that $u = U(y)$ is an example of a bounded linear operation $x = B(y)$ which maps $C(\Gamma)$ into $C(\overline{D})$ with the properties (6), (7) and (8).

It is easy to see that $H(\overline{D})$ itself is an (M)-space with respect to the same partial ordering as $C(\overline{D})$, and further that $H(\overline{D})$ is isometric and lattice isomorphic with $C(\Gamma)$. But it is to be noticed that the $\sup(x_1, x_2)$ of x_1 and x_2 in $H(\overline{D})$ does not necessarily coincide with the $\sup(x_1, x_2)$ of x_1 and x_2 in $C(\overline{D})$.

There are many ways of obtaining $u = U(y) \in H(\overline{D})$ from a given $y(\zeta) \in C(\Gamma)$. The well-known sweeping out process proceeds as follows: Let $\{K_n \mid n = 1, 2, \ldots\}$ be a sequence of circular domains $K_n = K(\zeta_n, r_n) = \{\zeta \mid |\zeta - \zeta_n| < r_n\}$ with the centers ζ_n and the radii r_n, completely contained in D (i.e. the closure $\overline{K}_n$ of K_n is contained in D) with the property:

(9) *for any* $\zeta_0 \in D$, *there exists an* $r_0 > 0$ *such that* $K(\zeta_0, r_0) \subset K_n$ *for infinitely many* n.

For each n, let us define a bounded linear operation $x' = P_n(x)$ which maps $C(\overline{D})$ into itself by the following conditions:

(10) $\quad x'(\zeta)$ *is harmonic in* K_n,

(11) $\quad x'(\zeta) \equiv x(\zeta)$ *on* $\overline{D} - K_n$.

It is then easy to see that $P_n(x)$ satisfies the following conditions:

(12) $\quad x \geqq 0$ *implies* $P_n(x) \geqq 0,$

(13) $\quad P_n(x) = x$ *if and only if* $x(\zeta)$ *is harmonic in* K_n,

(14) $\quad \| P_n(x) \| = \| x \|.$

From (9) and (13) follows:

(15) $\quad P_n(x) = x,\ n = 1, 2, \ldots,$ *if and only if* $x(\zeta)$ *is harmonic in* D.

In terms of these linear operations $P_n(x)$, we may state the fundamental result of the sweeping out process as follows:

Theorem 1. Let D be a bounded domain in the Gaussian plane which is regular for the Dirichlet problem. Let Γ be the boundary of D. For any $y(\zeta) \in C(\Gamma)$, let $x(\zeta)$ be any continuous extension of $y(\zeta)$ from Γ to $\overline{D}$. If *we put* $x_n = P_n P_{n-1} \cdots P_1(x)$, $n=1, 2, \ldots$, *then the sequence* $\{x_n \mid n=1, 2, \ldots\}$ *converges strongly in* $C(\overline{D})$ (i.e. *uniformly on* $\overline{D}$) *to the solution* $u(\zeta)$ *of the Dirichlet problem for the domain* D *and the boundary value* $y(\zeta)$. *In other words, the sequence of bounded linear operations* $\{Q_n \mid n=1, 2, \ldots\}$, *where* $Q_n = P_n P_{n-1} \cdots P_1$, $n=1, 2, \ldots$, *converges stronaly on* $C(\overline{D})$ *to the bounded linear operation* $V \equiv UA$.

It is not difficult to see that, by a slight modification of the argument used in the proof of Theorem 1, we may obtain

Theorem 2. Under the same assumptions as in Theorem 1, *let us put* $\tilde{x}_n = P_1 P_2 \cdots P_n(x)$, $n=1, 2, \ldots$, *for any* $x(\zeta) \in C(\overline{D})$. *Then the sequence* $\{\tilde{x}_n \mid n=1, 2, \ldots\}$ *converges strongly in* $C(\overline{D})$ (i.e. *uniformly on* $\overline{D}$) *to the same limit* $u = V(x) \equiv UA(x)$ *as in Theorem 1. In other words, the sequence of bounded linear operations* $\{\tilde{Q}_n \mid n=1, 2, \ldots\}$, *where* $\tilde{Q}_n = P_1 P_2 \cdots P_n$, $n=1, 2, \ldots$, *converges strongly on* $C(\overline{D})$ *to the bounded linear operation* $V \equiv UA$.

5. Let D be the same as in § 4. For any $\zeta_0 \in D$, let us denote by $\rho(\zeta_0)$ the distance of ζ_0 from the boundary Γ of D. For any $x(\zeta) \in C(\overline{D})$, let $x'(\zeta)$ be an element of $C(\overline{D})$ which is uniquely determined by the following conditions:

(16) *if* $\zeta_0 \in D$, *then* $x'(\zeta_0)$ *is the mean value of* $x(\zeta)$ *in the circular domain* $K(\zeta_0, {}^1/_2\, \rho(\zeta_0))$,

(17) $x'(\zeta) \equiv x(\zeta)$ *on* Γ.

Then $x' = R(x)$ is a bounded linear operation which maps $C(\overline{D})$ into itself and satisfies:

(18) $x \geqq 0$ *implies* $R(x) \geqq 0$,

(19) $R(x) = x$ *if and only if* $x(\zeta) \in H(D)$,

(20) $\| R(x) \| = \| x \|$.

By a similar argument as in the proof of Theorems 1 and 2, we may obtain

Theorem 3. Let D *be a bounded domain in the Gaussian plane which is regular for the Dirichlet problem. Let* Γ *be the boundary of* D. *For any* $y(\zeta) \in C(\Gamma)$, *let* $x(\zeta)$ *be any continuous extension of* $y(\zeta)$ *from* Γ *to* $\overline{D}$. *If we put* $x_n = R^n(x)$, $n=1, 2, \ldots$, *then the sequence* $\{x_n \mid n=1, 2, \ldots\}$ *converges strongly in* $C(\overline{D})$ (i.e. *uniformly on* $\overline{D}$) *to the solution* $u(\zeta)$ *of the Dirichlet problem for the domain* D *and the boundary value* $y(\zeta)$. *In other words, the sequence of the iterations* $\{R^n \mid n=1, 2, \ldots\}$ *converges strongly in* $C(\overline{D})$ *to the bounded liner operation* $V \equiv UA$.

6. Let D be the same as in § 4. Let ζ_0 be an arbitrary point of $\overline{D}$. For any $x(\zeta) \in C(\overline{D})$, put

(21) $$f(\zeta_0, x) = u(\zeta_0),$$

where $u=V(x)$ is the solution of the Dirichlet problem for the domain D corresponding to the boundary value $y=A(x)$. Then $f(\zeta_0, x)$ is a bounded linear functional defined on the (M)-space $C(\overline{D})$ with the properties:

$$x\geqq 0 \ \textit{implies} \ f(\zeta_0, x)\geqq 0, \tag{22}$$

$$x\equiv 1 \ \textit{implies} \ f(\zeta_0, x)=1. \tag{23}$$

Hence[4] there exists a countably additive measure $P(\zeta_0, E)$ defined for all Borel subsets E of $\overline{D}$ such that $P(\zeta_0, \overline{D})=1$ and

$$u(\zeta_0)\equiv f(\zeta_0, x)=\int_{\overline{D}} P(\zeta_0, d\zeta)x(\zeta) \tag{24}$$

for any $x(\zeta)\in C(\overline{D})$. Since $u(\zeta)=0$ on $\overline{D}$ if $x(\zeta)=0$ on Γ (i.e. if $y\equiv A(x)=0$), so we see that the mass distribution $P(\zeta_0, E)$ is distributed only on Γ. Thus $P(\zeta_0, E)$ is a countably additive measure defined for all Borel subsets E of Γ such that $P(\zeta_0, \Gamma)=1$ and

$$u(\zeta_0)=f(\zeta_0, x)=\int_{\Gamma} P(\zeta_0, d\zeta)y(\zeta) \tag{25}$$

for any $y(\zeta)\in C(\Gamma)$. It is clear that, for any $\zeta_0\in D$, the measure $P(\zeta_0, E)\equiv P(\zeta_0, E, D)$ thus obtained is nothing else than the harmonic measure[5] in the sense of R. Nevanlinna of a Borel subset E of Γ with respect to the domain D and the point ζ_0. If $\zeta_0\in\Gamma$, then the mass distribution $P(\zeta_0, E)$ is concentrated at ζ_0, i.e. $P(\zeta_0, E)=1$ or 0 according as $\zeta_0\in E$ or not.

For any $x(\zeta)\in C(\overline{D})$ and for each n, let us define $x'=P_n(x)\in C(\overline{D})$ by means of a circular domain $K_n=K(\zeta_n, r_n)$ as stated in § 4. It is then easy to see that, for any $\zeta_0\in D$, the value $x'(\zeta_0)$ of $x'(\zeta)$ at $\zeta=\zeta_0$ is obtained from $x(\zeta)$ by taking the integral:

$$x'(\zeta_0)=\int_{\overline{D}} P_n(\zeta_0, d\zeta)x(\zeta), \tag{26}$$

where $P_n(\zeta_0, E)$ is a countably additive measure defined for all Borel subsets E of $\overline{D}$ with the following properties: (i) $P_n(\zeta_0, E)$ is a mass distribution on the circumference $C_n=Bd(K_n)$ of K_n and is given by

$$P_n(\zeta_0, E)=\frac{1}{2\pi}\int_E \frac{d\theta}{r_n^2-2r_n\rho\cos(\theta-\varphi)+\rho^2}, \quad \zeta_0=\zeta_n+\rho e^{i\varphi}. \tag{27}$$

if $\zeta_0\in K_n$; (ii) $P_n(\zeta_0, E)$ is a mass distribution concentrated at ζ_0 if $\zeta_0\in\overline{D}-K_n$.

Let us put $Q_1(\zeta_0, E)=P_1(\zeta_0, E)$ and

4) S. Kakutani, loc. cit. (2).

5) R. Nevanlinna, Eindeutige analytische Funktionen, 1936.

(28) $$Q_n(\zeta_0, E)=\int_{\bar{D}} P_n(\zeta_0, d\zeta)Q_{n-1}(\zeta, E), \quad n=2, 3,\dots$$

Then it is easy to see that the value $x_n(\zeta_0)$ of $x_n=Q_n(x)=P_nP_{n-1}\cdots P_1(x)$ at $\zeta=\zeta_0$ is obtained from $x(\zeta)$ by taking the integral:

(29) $$x_n(\zeta_0)=\int_{\bar{D}} Q_n(\zeta_0, d\zeta)x(\zeta)\,.$$

Thus we may observe the sweeping out process as a non-homogeneous Markoff process in the space $C(\bar{D})$ of continuous functions $x(\zeta)$ defined on $\bar{D}$. From Theorem 1 follows :

Theorem 4. Let D be a bounded domain in the Gaussian plane which is regular for the Dirichlet problem. Let Γ be the boundary of D. Let us define the kernels $P_n(\zeta_0, E)$ as in above. Then, for any $\zeta_0 \in D$, in a non-homogeneous Markoff proces, in which the n-th transition probability is given by $P_n(\zeta_0, E)$, the sequence of composed kernels $\{Q_n(\zeta_0, E) \mid n=1, 2,\dots\}$ converges weakly to the harmonic measure $P(\zeta_0, E, D)$ in the sense of R. Nevanlinna of the set E with respect to the domain D and the point ζ_0, where the weak convergence means that, for any $x(\zeta) \in C(\bar{D})$ with $y=A(x)$,

(30) $$\int_{\bar{D}} Q_n(\zeta_0, d\zeta)x(\zeta)\to\int_{\bar{D}} P(\zeta_0, d\zeta, D)x(\zeta)=\int_{\Gamma} P(\zeta_0, d\zeta, D)y(\zeta)$$

as $n\to\infty$.

In the same way from Theorem 2 follows:

Theorem 5. Under the same assumptions as in Theorem 4, let us put $\tilde{Q}_1(\zeta_0, E)=P_1(\zeta_0, E)$ and

(31) $$\tilde{Q}_n(\zeta_0, E)=\int_{\bar{D}} Q_{n-1}(\zeta_0, d\zeta)P_n(\zeta, E), \quad n=2, 3,\dots$$

Then the sequence of composed kernels $\{\tilde{Q}_n(\zeta_0, E) \mid n=1, 2,\dots\}$ converges weakly to the harmonic measure $P(\zeta_0, E, D)$.

From the standpoint of the theory of the sweeping out process, Theorem 5 deserves more attention than Theorem 4. In fact, (31) means that the n-th mass distribution $\tilde{Q}_n(\zeta_0, E)$ is obtained from the (n-1)-th $\tilde{Q}_{n-1}(\zeta_0, E)$ by sweeping out the masses distributed inside K_n onto the boundary C_n of K_n according to the law given by $P_n(\zeta_0, E)$, while it is not so clear what the kernels $Q_n(\zeta_0, E)$ mean in Theorem 4.

We may also interprete Theorem 5 in the following way: Consider a Brownian motion $\{\zeta_0+(z(t, \omega)-z(0, \omega)) \mid -\infty<t<\infty, \omega \in \Omega\}$ starting from $\zeta_0 \in D$. As was shown in a preceding paper,[6] for any Borel subset E of the boundary Γ

6) S. Kakutani, Two-dimensional Brownian motion and harmonic functions, Proc. 20 (1944)

of D, the probability that the Brownian motion starting from ζ_0 will enter into E for some $t=t_\infty(\omega)\equiv\tau(\zeta_0, \Gamma, \omega)>0$ without entering into Γ-E before it, is equal to the harmonic measure $P(\zeta_0, E, D)$ in the sense of R. Nevanlinna of the set E with respect to the domain D and the point ζ_0.

Let us now define, for any $\omega \in \Omega$, the sequence of real numbers $\{t_n(\omega) \mid n=1, 2,\dots\}$ as follows: $t_1(\zeta)=$ the smallest value of t for which $t>0$ and $\zeta_0+(z(t, \omega)-z(0, \omega)) \in C_1=Bd(K_1)$ if $\zeta_0 \in K_1$; $t_1(\omega)=0$ if $\zeta_0 \in D-K_1$. In case $t_{n-1}(\omega)$ is already defined, $t_n(\omega)=$ the smallest value of t for which $t>t_{n-1}(\omega)$ and $\zeta_0+(z(t, \omega)-z(0, \omega)) \in C_n=Bd(K_n)$ if $\zeta_0+(z(t_{n-1}(\omega), \omega)-z(0, \omega)) \in K_n$; $t_n(\omega)=t_{n-1}(\omega)$ if $\zeta_0+(z(t_{n-1}(\omega), \omega)$-$z(0, \omega)) \in D$-$K_n$. Then it is easy to see that $\{t_n(\omega) \mid n=1, 2,\dots\}$ is a monotone non-decreasing sequence of ω-measurable functions of ω such that

$$(32) \qquad \lim_{n\to\infty} t_n(\omega)=t_\infty(\omega)$$

almost everywhere on Ω, and consequently that

$$(33) \qquad \begin{aligned} &\lim_{n\to\infty}(\zeta_0+(z(t_n(\omega), \omega)-z(0, \omega))) \\ &=\zeta_0+(z(t_\infty(\omega), \omega)-z(0, \omega))=\alpha(\zeta_0, \Gamma, \omega) \end{aligned}$$

almost everywhere on Ω, where $\alpha(\zeta_0, \Gamma, \omega)$ denotes the point of Γ at which the Brownian motion starting from ζ_0 enters into Γ for the first time after $t=0$. Further it is not difficult to see that the mass distribution $\tilde{Q}_n(\zeta_0, E)$ is obtained from the mesurable function $t_n(\omega)$ by the formula:

$$(34) \qquad Q_n(\zeta_0, E)=Pr\{\omega \mid \zeta_0+(z(t_n(\omega), \omega)-z(0, \omega)) \in E\},$$

where the right hand side means the probability (=measure) of the set of all $\omega \in \Omega$ such that $\zeta_0+(z(t_n(\omega), \omega)-z(0, \omega)) \in E$. From these follows easily that the sequence $\{Q_n(\zeta_0, E) \mid n=1, 2,\dots\}$ converges weakly to the harmonic measure $P(\zeta_0, E, \overline{D})$, or in other words, for any $x(\zeta) \in C(\overline{D})$, the sequence

$$(35) \qquad \int_{\overline{D}} Q_n(\zeta_0, d\zeta)x(\zeta)=\int_{\Omega} x(\zeta_0+(z(t_n(\omega), \omega)-z(0, \omega)))d\omega,$$

$n=1, 2,\dots$, converges to

$$(36) \qquad \begin{aligned} \int_{\overline{D}} P(\zeta_0, d\zeta, \overline{D})x(\zeta)&=\int_{\Omega} x(\zeta_0+(z(t_\infty(\omega), \omega)-z(0, \omega)))d\omega \\ &=\int_{\Omega} x(\alpha(\zeta_0, \Gamma, \omega))d\omega \end{aligned}$$

as $n\to\infty$.

7. An analogous situation holds for the case of Theorem 3. In this case, the value $x'(\zeta_0)$ of $x'=R(x)$ at $\zeta=\zeta_0$ is obtained from $x(\zeta)$ by taking the integral:

$$(37) \qquad x'(\zeta_0)=\int_{\overline{D}} R(\zeta_0, d\zeta)x(\zeta),$$

where the kernel $R(\zeta_0, E)$ is a countably additive measure defined for all Borel subsets E of $\overline{D}$ and is given by

$$(38) \qquad R(\zeta_0, E)=\frac{\text{measure of } E\cap K\left(\zeta_0, \frac{1}{2}\rho(\zeta_0)\right)}{\text{measure of } K\left(\zeta_0, \frac{1}{2}\rho(\zeta_0)\right)}$$

if $\zeta_0\in D$, and $R(\zeta_0, E)$ is a mass distribution concentrated at ζ_0 if $\zeta_0\in\Gamma$.

It is easy to see that that the value $x_n(\zeta_0)$ of $x_n=R^n(x)$ at $\zeta=\zeta_0$ is obtained from $x(\zeta)$ by taking the integral:

$$(39) \qquad x_n(\zeta_0)=\int_{\overline{D}} R^{(n)}(\zeta_0, d\zeta)x(\zeta),$$

where $R^{(1)}(\zeta_0, E)=R(\zeta_0, E)$ and

$$(40) \qquad R^{(n)}(\zeta_0, E)=\int_{\overline{D}} R(\zeta_0, d\zeta)R^{(n-1)}(\zeta, E), \quad n=2, 3,\dots$$

From Theorem 3 then follows:

Theorem 6. Let D be a bounded domain in the Gaussian plane which is regular for the Dirichlet problem. Let $R(\zeta_0, E)$ be defined as in above. Then, in a homogeneuos Markoff process in which the transition probability is given by $R(\zeta_0, E)$, the sequence of iterated kernels $\{R^{(n)}(\zeta_0, E)\,|\,n=1,2,\dots\}$ converges weakly to the harmonic measure $P(\zeta_0, E, D)$ in the sense of R. Nevanlinna of the set E with respect to the domain D and the point ζ_0, where the weak convergence means the same as in Theorem 4.

8. Let us now consider the case when D is an arbitrary bounded domain in the Gaussian plane which is not necessarily regular. In this case we cannot say that, for any $x(\zeta)\in C(\overline{D})$, the sequence $\{Q_n(x)\,|\,n=1, 2,\dots\}$, $\{\tilde{Q}_n(x)\,|\,n=1, 2,\dots\}$ or $\{R^n(x)\,|\,n=1, 2,\dots\}$ converges strongly in $C(\overline{D})$ (i.e. uniformly on $\overline{D}$). But it will be easily seen that these sequences converge at every point of D and that the convergence is even uniform on every compact set contained in D. Further, this limit function is nothing else than the generalized solution of the Dirichlet problem in the sense of N. Wiener[7] for the domain D which depends only on the boundary value $y=A(x)$ of $x(\zeta)$ on Γ.

7) N. Wiener, Certain notions in potential theory, Journ. of Math. and Phys., M.I.T., 3(1923).