PAPERS COMMUNICATED:

1. *On Canonical Transformations.*[*)]

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§ 1. *Introduction.*

It is known that a contact transformation does not change the form of a Hamiltonian canonical system of differential equations. Transformations not changing the canonical form are generally called *canonical.* For the transformations to be canonical, however, it is not necessary to be contact. Although extension of the contact transformations has been made,[**)] it does not include all the canonical transformations. In the present note the necessary and sufficient condition for canonicity is obtained purely algebraically and theorems on canonical transformations especially the necessary and sufficient condition for the existence of linear canonical but not contact transformations for a Hamiltonian canonical system are proved.

Let

$$\dot{q}_i \equiv \frac{d}{dt} q_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i \equiv \frac{d}{dt} p_i = -\frac{\partial H}{\partial q_i}, \quad (i = 1, \ldots, n) \tag{1}$$

be a given canonical system, where H is assumed not to contain the variable t explicitly.

We transform the variables q, p to Q, P by the equations

$$Q_i = Q_i(q, p), \quad P_i = P_i(q, p), \quad (i = 1, \ldots, n), \tag{C}$$

and assume that the functions Q, P do not contain t and, for the convenience of explanation, we shall use the matrix-, tensor- and vector-notations (The dummy indices of tensors run over from 1 to n.)

Writing

$$\mathfrak{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}, \quad \mathfrak{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \quad \mathfrak{Q} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix}, \quad \mathfrak{P} = \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix};$$
$$\mathfrak{h}_1 = \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_n} \end{pmatrix}, \quad \mathfrak{h}_2 = \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \end{pmatrix}, \quad \mathfrak{H}_1 = \begin{pmatrix} \frac{\partial H}{\partial Q_1} \\ \vdots \\ \frac{\partial H}{\partial Q_n} \end{pmatrix}, \quad \mathfrak{H}_2 = \begin{pmatrix} \frac{\partial H}{\partial P_1} \\ \vdots \\ \frac{\partial H}{\partial P_n} \end{pmatrix}, \tag{2}$$

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) S. Lie, *Die Allgemeinste Berührungstransformationen,* Gesammelte Abhandlungen **3, 295.

we have the canonical system (1) in the form

(3) $$\begin{pmatrix}\dot{\mathfrak{q}}\\ \dot{\mathfrak{p}}\end{pmatrix}=\begin{pmatrix}\mathfrak{h}_2\\ -\mathfrak{h}_1\end{pmatrix}.$$

If we write

(4) $$\begin{pmatrix}\frac{\partial Q_1}{\partial q_1}\cdots\cdots\frac{\partial Q_1}{\partial q_n}\\ \cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\\ \frac{\partial Q_n}{\partial q_1}\cdots\cdots\frac{\partial Q_n}{\partial q_n}\end{pmatrix}=A=(a_i{}^j),\quad \left(\frac{\partial Q_i}{\partial p_j}\right)=B(b_i{}^j),\quad \begin{matrix}\left(\frac{\partial P_i}{\partial q_j}\right)=C=(c_i{}^j),\\ \left(\frac{\partial P_i}{\partial p_j}\right)=D=(d_i{}^j),\end{matrix}$$

the vectors $\begin{pmatrix}\dot{\mathfrak{q}}\\ \dot{\mathfrak{p}}\end{pmatrix}$ and $\begin{pmatrix}\mathfrak{h}_1\\ \mathfrak{h}_2\end{pmatrix}$ being contra-variant and co-variant respectively, the transformed system of the canonical system (3) becomes

(5) $$\begin{pmatrix}\dot{\mathfrak{Q}}\\ \dot{\mathfrak{P}}\end{pmatrix}=\begin{pmatrix}A\ B\\ C\ D\end{pmatrix}\begin{pmatrix}D' & -B'\\ -C' & A'\end{pmatrix}\begin{pmatrix}\mathfrak{H}_2\\ -\mathfrak{H}_1\end{pmatrix},$$

where A' etc. denote the transposed matrices of A etc. Hence

(6) $$\begin{pmatrix}A\ B\\ C\ D\end{pmatrix}\begin{pmatrix}D' & -B'\\ -C' & A'\end{pmatrix}\begin{pmatrix}\mathfrak{H}_2\\ -\mathfrak{H}_1\end{pmatrix}=\begin{pmatrix}\mathfrak{H}_2\\ -\mathfrak{H}_1\end{pmatrix}$$

is necessary and sufficient for conserving the canonical form. Except when *det* $\begin{pmatrix}A\ B\\ C\ D\end{pmatrix}=0$, the equation (6) is equivalent to

(7) $$\left\{\begin{pmatrix}D' & -B'\\ -C' & A'\end{pmatrix}\begin{pmatrix}A\ B\\ C\ D\end{pmatrix}-E_{2n}\right\}\begin{pmatrix}\mathfrak{h}_2\\ -\mathfrak{h}_1\end{pmatrix}=0.$$

where E_m denotes the unit matrix of m-th degree.

A sufficient condition for (7):

(8) $$\begin{pmatrix}D' & -B'\\ -C' & A'\end{pmatrix}\begin{pmatrix}A\ B\\ C\ D\end{pmatrix}-E_{2n}=0$$

is, in *Lagrange's* bracket-expressions in Q, P-space, equivalent to

(9) $$[q_r,p_s]=\delta_{rs},\quad [q_r,q_s]=0,\quad [p_r,p_s]=0,\quad (r,s=1,\ldots,n).$$

Hence we obtain

Theorem 1: In order to conserve the form of a canonical system, it is sufficient that the transformation is contact. (8) *is the necessary and sufficient condition for the transformation* (C) *to be contact.*

§ 2. *Zero-divisors.*

We understand the vectors $\begin{pmatrix}\mathfrak{h}_2\\ -\mathfrak{h}_1\end{pmatrix}$, $\begin{pmatrix}\mathfrak{H}_2\\ -\mathfrak{H}_1\end{pmatrix}$ as singular matrices,*

$$\overbrace{\begin{pmatrix}\frac{\partial H}{\partial p_1}\,0\cdots\cdots 0\\ \cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\\ -\frac{\partial H}{\partial q_n}\,0\cdots\cdots 0\end{pmatrix}}^{2n}\Bigg\}2n,\qquad \begin{pmatrix}\frac{\partial H}{\partial P_1}\,0\cdots\cdots 0\\ \cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\\ -\frac{\partial H}{\partial Q_n}\,0\cdots\cdots 0\end{pmatrix}$$

* Schreier-Sperner, *Einführung in die Analytishe Geometrie und Algebra* **2**, 55.

Then we have from (7)

Theorem 2: The necessary and sufficient condition for the transformation (C) to be canonical is

$$\begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} - E_{2n} = \text{a zero-divisor of} \begin{pmatrix} \mathfrak{h}_2 \\ -\mathfrak{h}_1 \end{pmatrix}.$$

Except 0, the zero-divisors of the non-zero matrix $\begin{pmatrix} \mathfrak{h}_2 \\ -\mathfrak{h}_1 \end{pmatrix}$ always depend on the Hamiltonian function H. Hence we have

Theorem 3: Canonical transformations which do not depend on the Hamiltonian function H, are necessarily contact transformations.

Since

$$\begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} - E_{2n} = \begin{pmatrix} D'A - B'C - E_n & D'B - B'D \\ A'C - C'A & A'D - C'B - E_n \end{pmatrix}, \tag{10}$$

and

$$\begin{aligned} (D'A - B'C - E_n)' &= A'D - C'B - E_n, \\ (D'B - B'D)' &= -(D'B - B'D), \\ (A'C - C'A)' &= -(A'C - C'A), \end{aligned}$$

there exist only $n^2 + 2\left(\frac{1}{2}n(n-1)\right) = 2n^2 - n$ independent elements in the matrix (10). We shall put

$$\left.\begin{aligned} d_j{}^i a_j{}^k - b_j{}^i c_j{}^k - \delta^{ik} &= x^{ik}, \quad (x^{ik}) = X, \\ d_j{}^i b_j{}^k - b_j{}^i d_j{}^k &= y^{ik}, \quad (y^{ik}) = Y, \\ a_j{}^i c_j{}^k - c_j{}^i d_j{}^k &= z^{ik}, \quad (z^{ik}) = Z. \end{aligned}\right\} (i, k = 1, \cdots, n), \tag{11}$$

and

$$-\frac{\partial H}{\partial q_i} = \mu_i, \quad \frac{\partial H}{\partial p_i} = \lambda_i, \qquad (i = 1, \cdots, n),$$

then the equation (7) becomes

$$\left.\begin{aligned} \lambda_k x^{ik} + \mu_k y^{ik} &= 0, \\ \mu_k x^{ki} + \lambda_k z^{ik} &= 0, \end{aligned}\right\} (i = 1, \cdots, n); \tag{12}$$

especially for $n = 1$,

$$\lambda_1 x^{11} = 0, \quad \mu_1 x^{11} = 0.$$

Accordingly for $n = 1$ the equations (12) have only one solution $x^{11} = 0$, except the case in which

$$\frac{\partial H}{\partial q_1} = 0, \quad \frac{\partial H}{\partial p_1} = 0.$$

Theorem 4: For n = 1, the necessary and sufficient condition for canonicity is that the transformation (C) is contact.

§ 3. *Linear canonical transformations.*

If (C) is a linear transformation, then the matrix (10) is constant. Hence in order to have a canonical but not contact transformation, it is necessary that there exist functions $\nu^{(i)}(p, q)$ and constant vectors $\begin{pmatrix}\mathfrak{L}^{(i)}\\ \mathfrak{M}^{(i)}\end{pmatrix}$ satisfying

$$\begin{pmatrix}\mathfrak{h}_2\\ -\mathfrak{h}_1\end{pmatrix} = \sum_{i=1}^{l} \nu^{(i)} \begin{pmatrix}\mathfrak{L}^{(i)}\\ \mathfrak{M}^{(i)}\end{pmatrix}, \qquad l \leqq 2n-1. \tag{13}$$

We can easily prove the following

Lemma 1: The necessary and sufficient condition for the existence of functions $\nu^{(i)}(p, q)$ *and constant vectors* $\begin{pmatrix}\mathfrak{L}^{(i)}\\ \mathfrak{M}^{(i)}\end{pmatrix}$ *which suffice (13) is that we can write*

$$H(q_1, \cdots, q_n, p_1, \cdots, p_n) = K(X^{(1)}, X^{(2)}, \cdots, X^{(l)}),$$

with

$$X^{(i)} = \lambda_j^{(i)} p_j - \mu_j^{(i)} q_j, \qquad (i = 1, \cdots, l \leqq 2n-1),$$

where λ, μ *denote suitably chosen constants.*

We assume that the Hamiltonian function H satisfies the condition of Lemma 1, and that the functions $\nu^{(i)}(p, q)$ and the vectors $\begin{pmatrix}\mathfrak{L}^{(i)}\\ \mathfrak{M}^{(i)}\end{pmatrix}$ are linearly independent respectively, then (12) is equivalent to the system of equations

$$\left.\begin{aligned} &(14)_1 \qquad \lambda_k^{(j)} x^{ik} + \mu_k^{(j)} y^{ik} = 0, \\ &(14)_2 \qquad \mu_k^{(j)} x^{ki} + \lambda_k^{(j)} x^{ik} = 0, \end{aligned}\right\} (i = 1, \cdots, n; j = 1, \cdots, l).$$

Accordingly,

Lemma 2: The necessary and sufficient condition for the existence of a linear canonical but not contact transformation is that the Hamiltonian function H satisfies the condition of Lemma 1 and that there exist solutions x, y, z of the equations (14) which are not all zero, with $\det\begin{pmatrix}A & B\\ C & D\end{pmatrix} \neq 0$.

We write

$$\mathfrak{L}^{(i)} = \begin{pmatrix}\lambda_1^{(i)}\\ \vdots\\ \lambda_n^{(i)}\end{pmatrix}, \quad \mathfrak{M}^{(i)} = \begin{pmatrix}\mu_1^{(i)}\\ \vdots\\ \mu_n^{(i)}\end{pmatrix}, \qquad (i = 1, \cdots, l);$$

$$\begin{pmatrix}\mathfrak{L}^{(1)} & \mathfrak{L}^{(2)} \cdots \mathfrak{L}^{(l)}\\ \mathfrak{M}^{(1)} & \mathfrak{M}^{(2)} \cdots \mathfrak{M}^{(l)}\end{pmatrix} \equiv \mathfrak{D}.$$

For $l = 2n-1$, since the vectors $\begin{pmatrix}\mathfrak{L}^{(i)}\\ \mathfrak{M}^{(i)}\end{pmatrix}$ are linearly independent, the rank of $\mathfrak{D}$ is $2n-1$. If we erase a suitable row, say $(\mu_1^{(1)}, \mu_1^{(2)}, \cdots, \mu_1^{(2n-1)})$, the rank of the reduced matrix is still $2n-1$. This reduced matrix is nothing but the coefficient-matrix of the system of the equations $(14)_1$ for $i = 1$:

$$\lambda_1^{(j)} x^{11} + \lambda_2^{(j)} x^{12} + \cdots + \lambda_n^{(j)} x^{1n} + \mu_2^{(j)} y^{12} + \cdots + \mu_n^{(j)} y^{1n} = 0,$$
$$(j = 1, \cdots, 2n-1).$$

And the only solution of this system is $x^{1i} = 0$, $y^{1i} = 0$. $(i = 1, \cdots, n)$.

Next, we consider the equations $i = 2$ of $(14)_1$. Since $y^{21} = -y^{12} = 0$ because of the above consideration, it must be $x^{2i} = 0$, $y^{2i} = 0$, $(i = 1, \cdots, n)$ Repeating this process we can conclude that the only solution is

$$x^{ik} = 0, \quad y^{ik} = 0, \quad z^{ik} = 0, \qquad (i, k = 1, \cdots, n).$$

For $l = 2n-2$, by erasing two rows, for instance $(\lambda_i^{(1)}, \cdots, \lambda_i^{(2n-2)})$ and

$(\mu_j^{(1)}, \cdots, \mu_j^{(2n-2)})$, from the matrix $\mathfrak{D}$, we obtain a quadratic matrix of the $(2n-2)$-th degree: We write its determinant as $D(\lambda_i, \mu_j)$. Then we can prove that the system of values

$$x^{ij} = (-1)^{j-i} \quad D(\mu_i, \lambda_j),$$
$$y^{ij} = (-1)^{n+j-i} D(\mu_i, \mu_j),$$
$$z_{ij} = (-1)^{n+j-i} D(\lambda_i, \lambda_j),$$

is a solution of (14), which is not all zero.

For $l \leqq 2n-3$, by extending the proof for $l = 2n-2$, we can easily prove the existence of the solutions which are not all zero. Hence we obtain

Lemma 3: The necessary and sufficient condition for the existence of the solutions x, y, z which are not all zero, is $l \leqq 2n-2$, *if* $\begin{pmatrix} \mathfrak{h}_2 \\ -\mathfrak{h}_1 \end{pmatrix} \neq 0$.

Finally we can prove:

Lemma 4: The system of equations

(11) $\qquad D'A - B'C = E_n + X, \quad D'B - B'D = Y, \quad A'C - C'A = Z$

is soluble for A, B, C, D by taking a suitable set of X, Y, Z satisfying (12). Among these solutions there exists a set A, B, C, D with $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq 0$.

Let

$$D'B = \frac{Y}{2}, \quad B = E_n,$$

then we have

$$D' = \frac{Y}{2}, \quad C = \frac{Y}{2} A - (E_n + X)$$

(15) $$A'YA - \{A'(E_n + X) - (E_n + X')A\} = Z.$$

Conversely with a solution A of (15) we get $B = E_n$, $D' = \frac{Y}{2}$ and $C = \frac{Y}{2}A - (E_n + X)$ as the solution of (11)

We can prove the existence of the solution of (15) with $\det\{A'Y + (E_n + X)\} \neq 0$ (16) by the method of the mathematical induction.

By combining these Lemmas we obtain

Theorem 5: In the case $n \geqq 2$, *for the existence of a linear canonical but not contact transformation for the system*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad (i = 1; \cdots, n),$$

it is necessary and sufficient that we can select the constants $\lambda_j^{(i)}, \mu_j^{(i)}$ $(i = 1, \cdots, l \leqq 2n-2;\ j = 1, \cdots, n)$, *so that, by putting*

$$X^{(i)} = \lambda_j^{(i)} p_j - \mu_j^{(i)} q_j \qquad (i = 1, \cdots, l \leqq 2n-2),$$

we may have

$$H(q_1, \cdots, q_n, p_1, \cdots, p_n) = K(X^{(1)}, X^{(2)}, \cdots, X^{(l)}).$$