62. Note on Irreducible Rings.

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The purpose of the present work(1) is to extend, partly, the well-known beautiful theory of simple algebras and their relationship with subalgebras(2) to irreducible rings; A ring we call *irreducible*, or *right-irreducible* to be precise, when it has a faithful irreducible right-module. More generally we call a ring *(right-)semi-irreducible*, when it has a faithful completely reducible right-module.(3) If an (irreducible) ring possesses a faithful irreducible right-ideal, then we speak of a *(right-) ideal-irreducible* ring. A *closed (right-) irreducible* ring is defined as a ring $\mathfrak{R}$ possessing a faithful irreducible right-module $\mathfrak{m}$ with $\mathfrak{R}$-endomorphism ring $\mathfrak{R}^*$, such that every $\mathfrak{R}^*$-endomorphism of $\mathfrak{m}$ is induced by $\mathfrak{R}$. Similarly defined are *(right-) ideal-semi-irreducible* and *closed (right-) irreducible* rings.

Let $\mathfrak{R}$ be a (right-) ideal-semi-irreducible ring, $\mathfrak{r}_1$ a faithful completely reducible right-ideal in $\mathfrak{R}$. Take one representative from each class of mutually isomorphic irreducible right subideals of $\mathfrak{r}_1$. The (restricted direct) *sum* $\mathfrak{r}_0$ *of the tatality of such representatives* is also a faithful completely reducible right ideal. Now we have:

Every faithful right-module of $\mathfrak{R}$ possesses a submodule isomorphic to $\mathfrak{r}_0$. In particular, $\mathfrak{r}_0$ is a minimal faithful right ideal in $\mathfrak{R}$. Every non-zero right-ideal of $\mathfrak{R}$ contains an irreducible right subideal, which is isomorphic with an irreducible component of $\mathfrak{r}_0$. A right-ideal of $\mathfrak{R}$ is irreducible if and only if it is generated by a primitive idempotent element. The sum of all (irreducible) right-ideals isomorphic with an irreducible right-ideal is an irreducible two-sided ideal, and every irreducible two-sided ideal is obtained in such manner. Every non-zero two-sided ideal contains an irreducible two-sided ideal. The (restricted direct) sum of all irreducible two-sided ideal, that is, the largest completely reducible two-sided ideal in $\mathfrak{R}$, is the smallest right-(as well as two-

(1) A fuller account is given in a forthcoming joint paper by G. Azumaya and the writer.

(2) Of R. Brauer, E. Noether and A. A. Albert, among others.

(3) For C. Chevalley's principal theorem of semi-irreducible ring, in the effect to embed a semi-irreducible ring densely in a closed one (in the sense of the weak topology of mappings in the (discrete) module, see T. Nakayama, Ueber einfache distributive Systeme unendlicher Ränge, these Proc. 20 (1944), Anhang.

sided) faithful two-sided ideal, and is by itself an ideal-iemi-irreducible ring. In the particular case of (ideal-) irreducible $\mathfrak{R}$ this last is the smallest (non-zero) two-sided ideal.

To prove these, let first $\mathfrak{m}$ be an arbitrary faithful right-module of $\mathfrak{R}$, and $\mathfrak{s}$ an irreducible subideal of $\mathfrak{r}_0$. Then $\mathfrak{m}\mathfrak{s}\neq 0$, whence $u\mathfrak{s}\neq 0$ with an element u in $\mathfrak{m}$, and this non-zero submodule of $\mathfrak{m}$ is isomorphic with $\mathfrak{s}$. The sum of such irreducible submoduli, $\mathfrak{s}$ running over all irreducible components in $\mathfrak{r}_0$, forms a submodule of $\mathfrak{m}$ isomorphic with $\mathfrak{r}_0$. Let $\mathfrak{t}$ be a second irreducible component of $\mathfrak{r}_0$, different from $\mathfrak{s}$. Then $\mathfrak{t}\mathfrak{s} = 0$, since it is contained in $\mathfrak{t}$ and is, on the other hand, a sum of subideals isomorphic to $\mathfrak{s}$. Then $\mathfrak{s}^2 = \mathfrak{s}$, because if $\mathfrak{s}^2 = 0$ we would have $\mathfrak{r}_0\mathfrak{s} = 0$. That (the idempotent irreducible right ideal) $\mathfrak{s}$ is generated by a primitive idempotent element can be seen as usual. The sum $\mathfrak{z}_{\mathfrak{s}}$ of all right ideals isomorphic to $\mathfrak{s}$ is a two sided ideal, and in fact $\mathfrak{s} = \mathfrak{R}\mathfrak{s}$, since every right ideal isomorphic to $\mathfrak{s}$ has a form $a\mathfrak{s}$ $(a \varepsilon \mathfrak{R})$. $c\mathfrak{R} \neq 0$ for every $c \neq 0$ from $\mathfrak{z}_{\mathfrak{s}}$; observe that the left annihilator of $\mathfrak{R}$ in $\mathfrak{z}_{\mathfrak{s}}$ is a right (in fact, two sided) ideal, which would contain a subideal isomorphic with $\mathfrak{s}$ if $\neq 0$. So $\mathfrak{R}c\mathfrak{R} = \mathfrak{z}_{\mathfrak{s}}$, which proves the two-sided irreducibility of $\mathfrak{z}_{\mathfrak{s}}$. $\mathfrak{z} = \mathfrak{R}\mathfrak{r}_0$ is the sum of all such $\mathfrak{z}_{\mathfrak{s}}$'s, and $\mathfrak{z}_{\mathfrak{s}}$, $\mathfrak{z}_{\mathfrak{t}}$ with non-isomorphic $\mathfrak{s}$, $\mathfrak{t}$ are orthogonal. Let $\mathfrak{a}$ be a non zero two sided ideal in $\mathfrak{R}$. $\mathfrak{z} \geqq \mathfrak{R}\mathfrak{r}_0\mathfrak{a} \geqq \mathfrak{r}_0{}^2 a = \mathfrak{r}_0\mathfrak{a} \neq 0$. Thus $\mathfrak{R}\mathfrak{r}_0\mathfrak{a}$ is, as a non zero two-sided subideal of $\mathfrak{z}$, a non-void sum of certain $\mathfrak{z}_{\mathfrak{s}}$'s. As $\mathfrak{R}\mathfrak{r}_0\mathfrak{a} \leqq a$, we conclude that $\mathfrak{a}$ contains at least one $\mathfrak{z}_{\mathfrak{s}}$. It follows then that $\mathfrak{z}$ is (not only right, but also) left faithful; for, if the left annihilator of $\mathfrak{z}$ which is a two sided ideal, were non-zero then it would contain a certain $\mathfrak{z}_{\mathfrak{s}}$, contrary to $\mathfrak{z}_{\mathfrak{s}}\mathfrak{z} = \mathfrak{z}_{\mathfrak{s}} \neq 0$. Consider then an arbitrary non zero right ideal $\mathfrak{r}$. $\mathfrak{r}\mathfrak{z} \neq 0$, whence $\mathfrak{r}\mathfrak{z}_{\mathfrak{s}} \neq 0$ with a certain $\mathfrak{z}_{\mathfrak{s}}$, $\mathfrak{r}$ contains then a subideal isomorphic with $\mathfrak{s}$. Now, a right principal ideal generated by a primitive idempotent is then irreducible, since every non-zero right ideal contains an irreducible subideal.

Further, *a right-ideal- (semi-) irreducible ring is always left-ideal (semi-) irreducible too.*[4] Namely, a left ideal $\mathfrak{R}e$ generated by a primitive idempotent e is also irreducible, because a left ideal $\mathfrak{R}a$ contains a non-zero idempotent if and only if $a\mathfrak{R}$ contains a such; The e istence of an x with $xax = x$ is necessary and sufficient for both. Take e from each of mutually non-isomophic irreducible right-ideals $\mathfrak{s}$. Then the (restricted direct) sum of $\mathfrak{R}e$'s is a faithful completely reducible left ideal.

A quasifield $\mathfrak{K}$ inverse-isomorphic to the endomorphism quasifield $\mathfrak{K}'$ of an up to isomorphism unique faithful irreducible right-module $\mathfrak{m}$ of an ideal-

(4) The irreducible case was communicated to me by G. Azumaya.

irreducible ring $\mathfrak{R}$ is said to belong to $\mathfrak{R}$. The rank of $\mathfrak{m}$ over $\mathfrak{K}'$ is called the dimension of $\mathfrak{R}$, and is denoted by $[\mathfrak{R}]$. A closed (right-) irreducible ring is ideal-irreducible, and is nothing but the full row-finite $[\mathfrak{R}]$-dimensional matric ring over its quasifield $\mathfrak{K}$. Its automorhism-class-group is isomorphic with that of the quasifield $\mathfrak{K}$. Further, a closed semi-irreducible ring is a complete (=non-restricted) direct sum of mutually orthogonal closed irreducible subrings, corresponding to non-isomorphic irreducible components (or, what is the same, corresponding to components in the ideal-decomposition) of the faithful completely reducible right-module. So, in the sequel we shall rather restrict ourselves to the irreducible case, as is usually done in the theory of semi-simple algebras also, for the sake of simplicity.

Let $\mathfrak{R}$ be a closed irreducible ring with center Z, and $\mathfrak{S}$ be a (nonnilpotent) simple algebra (=hypercomplex ring of a finite rank) over Z. The direct product $\mathfrak{R} \times \mathfrak{S}$ over Z is a closed irreducible ring with quasifeld isomorphic to that belonging to the simple ring (with chain condition) $\mathfrak{K} \times \mathfrak{S}$, where $\mathfrak{K}$ denotes the quasiffeld belonging to $\mathfrak{R}$. If $\mathfrak{z}$ is the smallest two-sided ideal of $\mathfrak{R}$, then $\mathfrak{z} \times \mathfrak{S}$ is that of $\mathfrak{R} \times \mathfrak{S}$. (If $\mathfrak{R}$ is, generally, a general irreducib.e ring, we may embed it into its closure with respect to a faithful irreducible right-module and thus construct its direct product with a hypercomplex system over the center of its closure. The direct product of $\mathfrak{R}$ with a simple algebra is then irreducible. If $\mathfrak{R}$ is ideal-irreducible, so is the product and the above assertion concerning smallest two-sided ideals remains valid too.)

Let in particular $\mathfrak{R}$ be a simple subring of (the closed irreducible ring) $\mathfrak{R}$ containing Z and of finite rank over Z. The commuter ring $V_{\mathfrak{R}}(\mathfrak{S})$ of $\mathfrak{S}$ in $\mathfrak{R}$ is a closed irreducible ring and the quasiffeld belonging to it is isomorphic with that belonging to the direct product $\mathfrak{R} \times \mathfrak{S}'$ (over Z), where $\mathfrak{S}'$ is an algebra inverse-isomorphic to $\mathfrak{S}$. The commuter ring of $V_{\mathfrak{R}}(\mathfrak{S})$ coincides with $\mathfrak{S}$; $V_{\mathfrak{R}}(V_{\mathfrak{R}}(\mathfrak{S})) = \mathfrak{S}$. $\mathfrak{S} \frown V_{\mathfrak{R}}(\mathfrak{S})$ is the common center of $\mathfrak{S}$ and $V_{\mathfrak{R}}(\mathfrak{S})$, and the product $\mathfrak{S}V\mathfrak{R}(\mathfrak{S})$ in $\mathfrak{R}$ is direct over it. Moreover this closed irreducible ring $\mathfrak{S}V\mathfrak{S}(\mathfrak{S})$ is the commuter ring of $\mathfrak{S} \frown V_{\mathfrak{R}}(\mathfrak{S})$ in $\mathfrak{R}$, and its smallest two-sided ideal is the (direct) product of that of $V_{\mathfrak{R}}(\mathfrak{S})$ with $\mathfrak{S}$. (If in particular $\mathfrak{B}$ is normal over Z then $\mathfrak{R} = \mathfrak{S} \times V_{\mathfrak{R}}(\mathfrak{S})$.) Every isomorphism of $\mathfrak{R}$ (with a second simple algebra) in = leaving Z elementwise fixed can be extended to an inner automorphism of $\mathfrak{R}$.

Now, let there be given a finite subgroup $\mathfrak{G} = \{E, S, \ldots, T\}$ of the automorphism-class-group of $\mathfrak{R}$, and let for every $S \in \mathfrak{G}$ a class representative S be given. A set of $(\mathfrak{G})^2$ regular elements $a_{S,T}$ of $\mathfrak{R}$ is called a factor set (belonging to $\mathfrak{G}$ and to the system $\{S\}$) when

i) $x^{\bar{S}\bar{T}} = a_{S,T}^{-1} x^{\overline{ST}} a_{S,T} \ (x \in \mathfrak{R};\ S,\ T \in \mathfrak{G})$,

ii) $a_{R,ST} a_{S,T} = a_{RS,T} a_{R,}^{\bar{T}} \ (R,\ S,\ T \in \mathfrak{G})$.

With a factor set we can introduce a *crossed product*

$$(\mathfrak{R},\ \mathfrak{G}) = \mathfrak{R}u_E + \mathfrak{R}u_S + \ldots + \mathfrak{R}u_T$$

in the usual manner, and this *is an ideal-irreducible ring with the smallest two-sided ideal* $(\mathfrak{z},\ \mathfrak{G})$, $\mathfrak{z}$ being such in $\mathfrak{R}$. *If in particular G is a finite group of outer automorphisms of $\mathfrak{R}$, the crossed product $(\mathfrak{R},\ G)$ with unit factor set $\{1\}$ is not only ideal-irreducible but closed.* And with its aid we can derive the following Galois theorem.(5)

The invariant system $\mathfrak{U}$ of G in $\mathfrak{R}$ is a closed irreducible ring, and G exhausts automorphisms of $\mathfrak{R}$ leaving $\mathfrak{U}$ elementwise fixed. $\mathfrak{R}$ has a linearly independent right-basis over $\mathfrak{U}$ consisting of (G) terms; in fact it has a normal right-basis over $\mathfrak{U}$.(6) Every closed irreducible subring of $\mathfrak{R}$ containing $\mathfrak{U}$ is the invariant system of a suitable subgroup of G. Thus the closed irreducible rings between $\mathfrak{U}$ and $\mathfrak{R}$ correspond 1–1 to the subgroups of G. The commuter ring of $\mathfrak{U}$ in $\mathfrak{R}$ is identical with the center Z of $\mathfrak{R}$, and the center K of $\mathfrak{U}$ is the invariant system of G in Z. The product $\mathfrak{U}Z$ in $\mathfrak{R}$ is direct over K, and the subgroup of G belonging to it consists of all elements in G leaving Z elementwise invariant. $\mathfrak{R}$ can be expressed as a row-finite full matric ring with G-invariant matric units over a simple ring with chain condition $\mathfrak{R}_0$, so that G may be looked upon as a group of outer automorphisms of $\mathfrak{R}_0$, and the invariant system $\mathfrak{U}_0$ of G in $\mathfrak{R}_0$ is a quasifield. $\mathfrak{U}$ is the matric ring over $\mathfrak{U}_0$ with the same system of matric units, and closed irreducible rings between $\mathfrak{R}$, $\mathfrak{U}$ correspond 1–1 to those (which are simple rings with chain condition) between $\mathfrak{R}_0$, $\mathfrak{U}_0$; the between-rings mutually corresponding belong to one and the same subgroup of G. Thus the Galois theory of $\mathfrak{R}/\mathfrak{U}$ is reduced to that of $\mathfrak{R}_0/\mathfrak{U}_0$.

These all may be shown in more or less similar manner as the well-established case of simple algebras (or of simple rings with chain condition). Particularly suited is an approach by G. Azumaya,(7) to whom also the pres-

(5) See N. Jacobson, The fundamental theorem of Galois theory for quasi-fields, An. Math. 41 (1940). Its extension to simple rings with chain condition, together with some refinements, has been given by G. Azumaya; see Azumaya, New foundation for the theory of simple rings, forthcoming in these Proc.

(6) Cf. Nakayama, Normal basis of a quasi-field, these Proc. 16 (1940).

(7) See Azumaya. l. c. 5). Its main feature is to embed the ring in the absolute endomorphism ring of its (faithful) right-module. It resembles thus with the methods of R. Brauer-H. Weyl and A. A. Albert, at least in the case of algebras, but is much smarter and directer.

Theorems 1, 2 in Azumaya's paper may be also transferred to close irreducible rings.

ent work owes much; his attendance, as well as his intervention and ciriticism, at the writer's lectures during the winter 1944–45, in which the most part of the present work was expounded, were so valuable and the writer wants to express here his best thanks to him.

Added in proof: The paper, On irreducible rings, referred to in (1) has appeared in Ann. Math. 48 (1947). Irreducible rings are called primitive rings in a paper by Jacobson appeared shortly before this joint paper by Azumaya and the writer; N. Jacobson, On the theory of primitive rings, Ann. Math. 48 (1947).

Meanwhile many papers have been made accessible to the writer which ought to have been referred to if had been known to him. Let the following two be particularly mentioned: E. Artin-G. Whaples, The theory of simple rings, Amer. J. Math. 65 (1943); N. Jacobson, Structure theory of simple rings without finiteness assumptions, Trans. Amer. Math. Soc. 57 (1945). The paper of Artin and Whaples is closely related to Azumaya's in (5), while in Jacobson's paper Chevalley's theorem, referred to in (3), was obtained independently.